The Uniform-Section Disk Spring

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The authors point out in this paper that initially coned annular-disk springs of uniform cross section may be proportioned to give a wide variety of load-deflection curves not readily obtainable with the more conventional forms of springs, and that, although the versatility of this type spring has long been indicated, the formulas available have not been presented in a manner to disclose readily the effect of spring proportions on characteristics. Therefore the authors have derived the formulas presented in this paper with the intention that the formulas will aid the designer in arriving at suitable characteristics by choice of spring geometry. These new formulas have been in use for several years at the General Motors Corporation research laboratories section, and their reliability has been checked by tests of springs used in a variety of special test equipment.

IN ADDITION to compactness along the axis of loading, the initially coned, annular-disk spring of uniform cross section may be proportioned to give a wide variety of load-deflection characteristics not readily obtainable with the more conventional forms of springs. By the simple expedient of varying the free cone height and the working range of deflections, spring rates may be varied from positive to zero to negative. The load-capacity and deflection range may be varied by the use of multiple springs arranged in series and/or parallel.

The versatility of the annular-disk spring has long been indicated. However, the formulas heretofore available have not been presented in a manner to disclose readily the effect of spring proportions on characteristics. Hence the designer could not make full use of this type of spring.

In this paper, it has been attempted to present formulas in a manner to aid the designer in arriving at suitable characteristics by choice of spring geometry. These new formulas have been in use for several years at the General Motors Corporation research laboratories section, and their reliability has been checked by tests of springs used in a variety of special test equipment. Experience has covered springs varying in outside diameter from 1 in. to 12 in., springs with ratios of outside to inside diameter from 1.4 to 5.5, and springs with ratios of free cone height to thickness giving practically the full range of characteristics plotted in Fig. 3. The present formulas are shown to be reliable for prediction of load-deflection curves. Lack of information on true stresses does not detract from their utility. Computed permissible maximum stresses for static loading are quite high, in the neighborhood of 220,000 lb per sq in., but experience indicates that these values may be used for design purposes when using plain carbon steel. In dynamic applications, fatigue tests are required.

Fig. 1 shows the type of load-deflection curve given by the diaphragm spring found in the ordinary oil can. It will be recalled that, as the oil-can bottom is deflected, we must at first exert considerable pressure and that, subsequently, the pressure required decreases in a manner similar to that shown in Fig. 1.

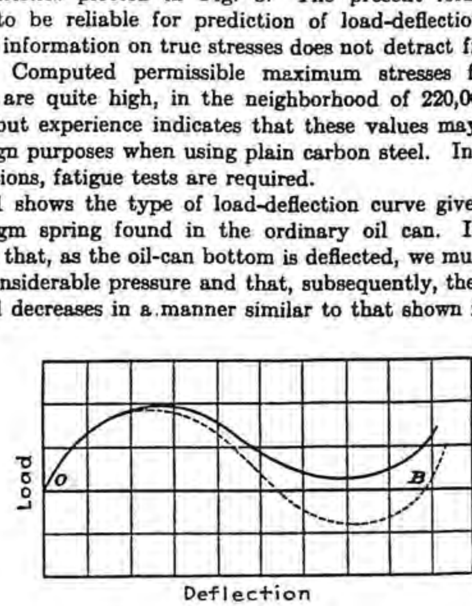

Fig. 1 General Shape of a Disk-Spring Load-Deflection Curve Having a Variable Spring Rate

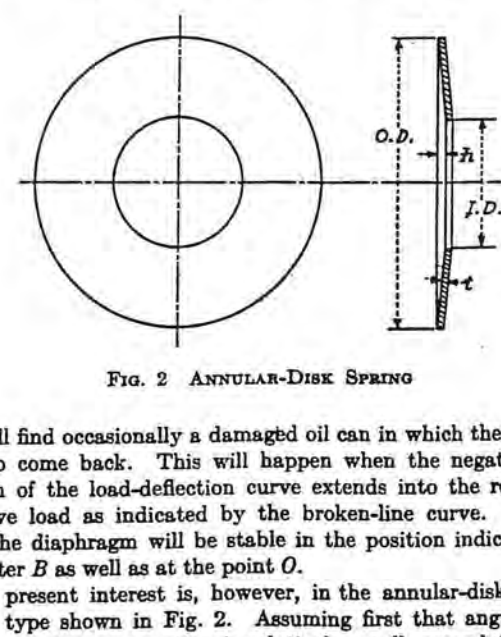

Fig. 2 Annular-Disk Spring

We will find occasionally a damaged oil can in which the bottom fails to come back. This will happen when the negative-rate portion of the load-deflection curve extends into the region of negative load as indicated by the broken-line curve. In this case, the diaphragm will be stable in the position indicated by the letter B as well as at the point O.

Our present interest is, however, in the annular-disk spring of the type shown in Fig. 2. Assuming first that angular deflection of the cross section is relatively small, second that the cross section remains undistorted in the deflected position, and third that loading and support are uniformly distributed around the respective circumferences, we obtain the following formulas for dished springs.

The formula for the load is

$$P = \frac{E\delta}{(1-\sigma^2)\,Ma^2}\left[\left(h-\delta\right)\left(h-\frac{\delta}{2}\right)t + t^3\right]$$

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The maximum stress in the upper edge is

$$S = \frac{E\delta}{(1-\sigma^2)\,M\,a^2}\left[C_1\left(h-\frac{\delta}{2}\right)+C_2 t\right]$$

The maximum stress in the lower edge is

$$S = \frac{E\delta}{(1-\sigma^2)\,M\,a^2}\left[C_1\left(h-\frac{\delta}{2}\right)-C_2 t\right]$$

where a = outside radius = half outside diameter, t = thickness, h = free height = height of truncated cone formed by the upper

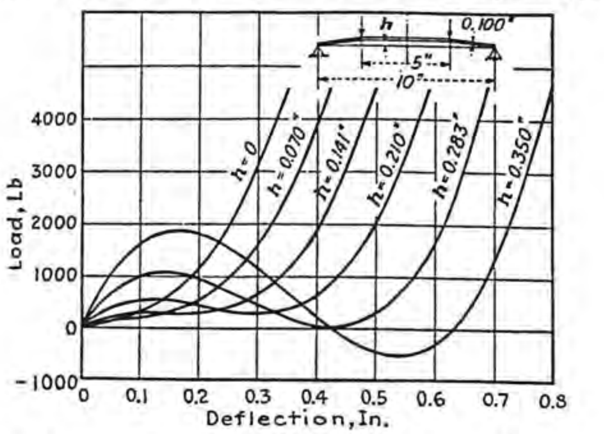

Fig. 3 Computed Characteristic Load-Deflection Curves of Six Disk Springs Identical Except for Free Height h

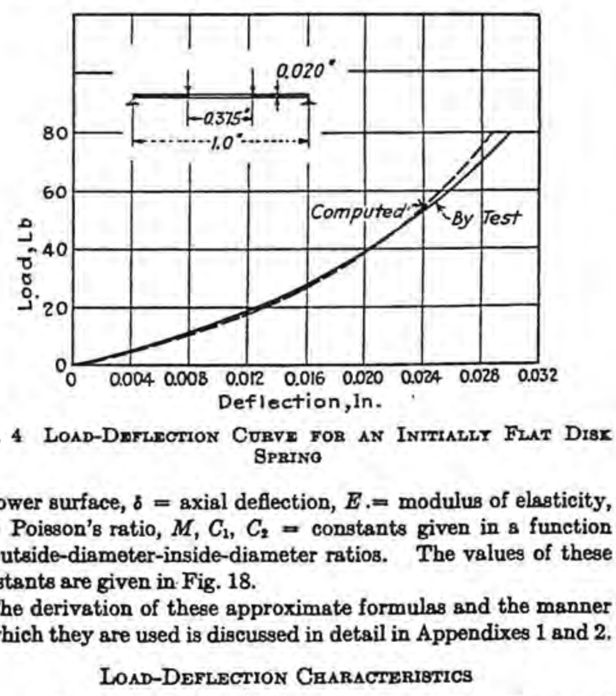

Fig. 4 Load-Deflection Curve for an Initially Flat Disk Spring

or lower surface, δ = axial deflection, E = modulus of elasticity, σ = Poisson's ratio, M, C_1, C_2 = constants given in a function of outside-diameter-inside-diameter ratios. The values of these constants are given in Fig. 18.

The derivation of these approximate formulas and the manner in which they are used is discussed in detail in Appendixes 1 and 2.

Load-Deflection Characteristics

Fig. 3 shows load-deflection curves calculated for a series of springs having the same diameter and thickness but varying in initial cone height h.

That these theoretical characteristics are obtainable in practice is shown by Figs. 4 to 7, inclusive, which cover outside diameters from 1 in. to $12^3/_4$ in.; outside-diameter-inside-diameter ratios from 1.7 to 4.25; and cone height-thickness ratios from 0 to 2.5. The agreement between theory and test is noteworthy.

In calculating the curves shown in Fig. 3, the load was assumed applied in the direction indicated. Another group of curves can be obtained from the same springs by applying the load in the opposite direction, as shown, for example, in Fig. 8.

Springs having load-deflection curves of the type h = 0.141 in. shown in Fig. 3 and the lower curve in Fig. 8, are often very useful inasmuch as they have a deflection range in which the load changes only very slightly, that is, a deflection range of

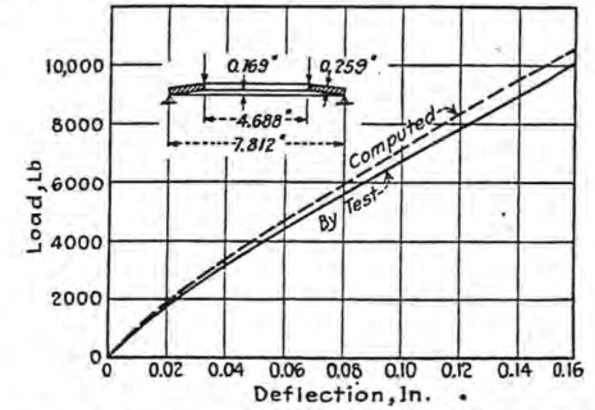

Fig. 5 Load-Deflection Curve of a Thick Disk Spring

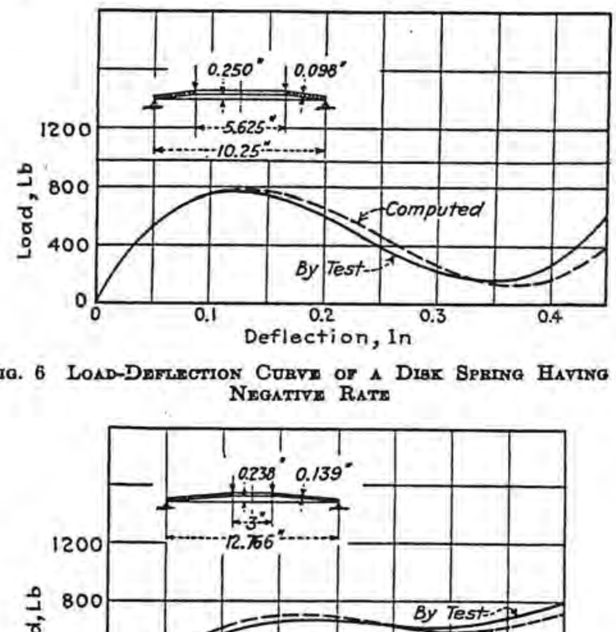

Fig. 6 Load-Deflection Curve of a Disk Spring Having a Negative Rate

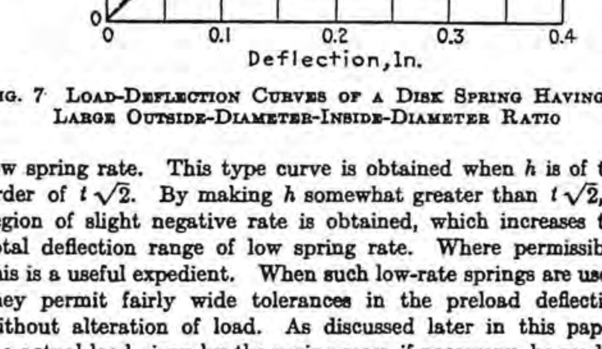

Fig. 7 Load-Deflection Curves of a Disk Spring Having a Large Outside-Diameter-Inside-Diameter Ratio

low spring rate. This type curve is obtained when h is of the order of $t\sqrt{2}$. By making h somewhat greater than $t\sqrt{2}$, a region of slight negative rate is obtained, which increases the total deflection range of low spring rate. Where permissible, this is a useful expedient. When such low-rate springs are used, they permit fairly wide tolerances in the preload deflection without alteration of load. As discussed later in this paper, the actual load given by the spring may, if necessary, be readily adjusted by alteration in the radial location of the support or point of loading.

An example of a machine design using such low-rate disk springs is shown in Fig. 9, wherein a live tailstock center is shown equipped with two disk springs in parallel arranged to take the thrust load of the bearings. The springs have a deflection range of 0.1 in. in which the load is practically constant, thus allowing for work expansion when long pieces are being machined without overloading the tailstock bearings.

Fig. 10 shows how the load capacity may be varied without alterations of any kind to the spring but by variation in the point of load application. As a first approximation, the load

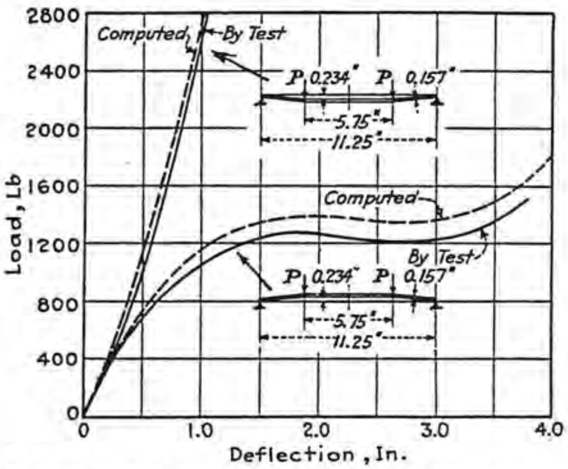

FIG. 8 LOAD-DEFLECTION CURVES OF A DISK SPRING LOADED IN TWO DIRECTIONS

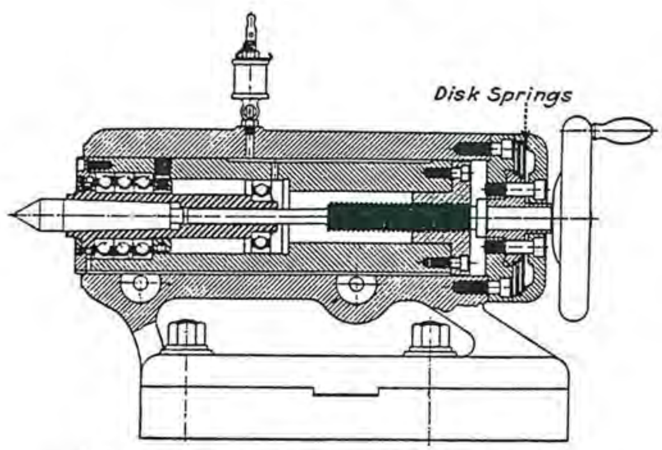

FIG. 9 TAILSTOCK CENTER WITH DISK SPRINGS

varies inversely as the length of the arm of the couple tending to rotate the cross section, and the deflection at the load varies directly as the length of the arm of the couple tending to rotate the cross section. The test curves shown in Fig. 10 show also the friction hysteresis loop resulting from the slight slip that occurs between the loading ring and the spring, and between the spring and the supporting ring, when these members are rigid. The width of the loop appears to bear no fixed relation to the load, as found also in other tests. Disk springs loaded in parallel have interspring friction, whereas springs loaded in series have friction at the loading and support points only. Hence, the amount of friction damping may be varied by a selection of series or parallel combinations.

Fig. 11 shows load-deflection curves obtained by stacking

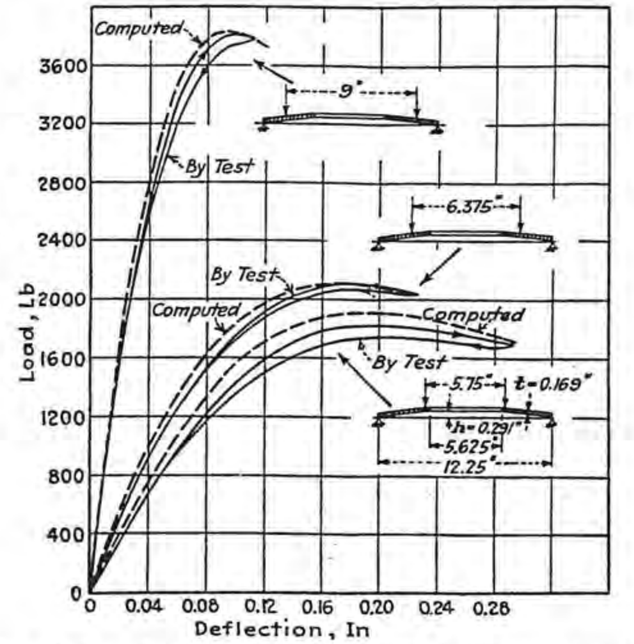

FIG. 10 LOAD-DEFLECTION CURVES OF A DISK SPRING LOADED AT VARIOUS DIAMETERS

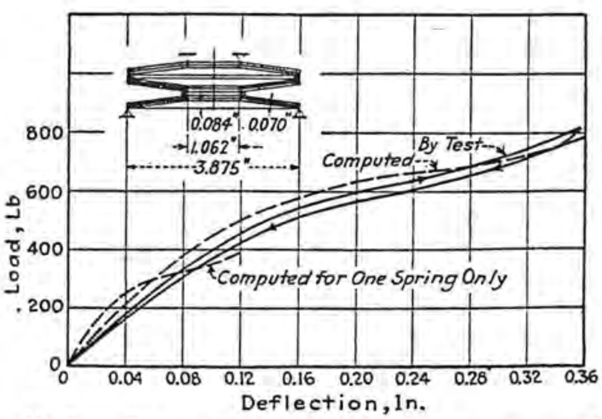

FIG. 11 LOAD-DEFLECTION CURVES AND HYSTERESIS OF SIX SPRINGS, THREE GROUPS OF TWO PARALLEL SPRINGS LOADED IN SERIES

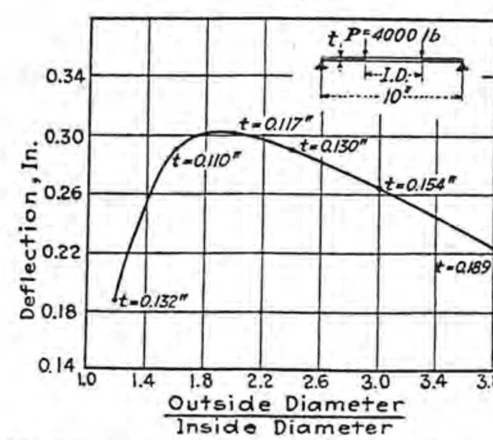

FIG. 12 INFLUENCE OF OUTSIDE-DIAMETER-INSIDE-DIAMETER RATIO ON THE FLEXIBILITY (DEFLECTION-LOAD) OF AN INITIALLY FLAT DISK SPRING UNDER 4000-LB LOAD

(The thickness of the spring has been varied to maintain a computed maximum stress of 200,000 lb per sq in.)

several springs in a series-parallel combination. As predicted by theory, the deflection range of the single spring is tripled and the load capacity is doubled.

The effect of the outside-diameter-inside-diameter ratios on the flexibility of disk springs is shown in Fig. 12. The curve in this figure considers an initially flat spring of given diameter

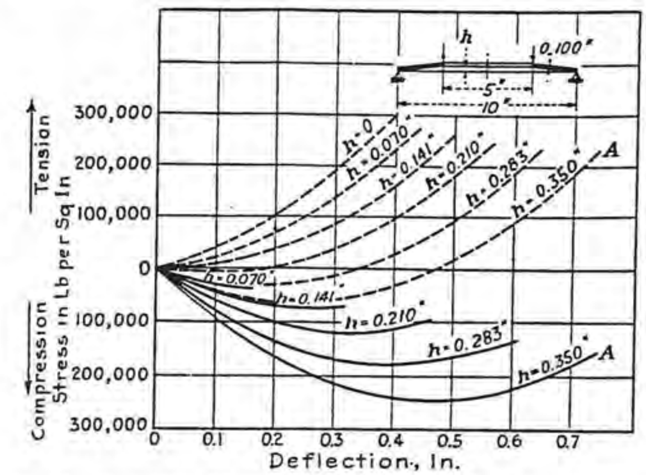

FIG. 13 COMPUTED MAXIMUM STRESSES FOR SIX DISK SPRINGS IDENTICAL EXCEPT FOR FREE HEIGHT

(Solid lines show stress at upper edge of inner diameter. Dashed lines show stress at lower edge of inner diameter.)

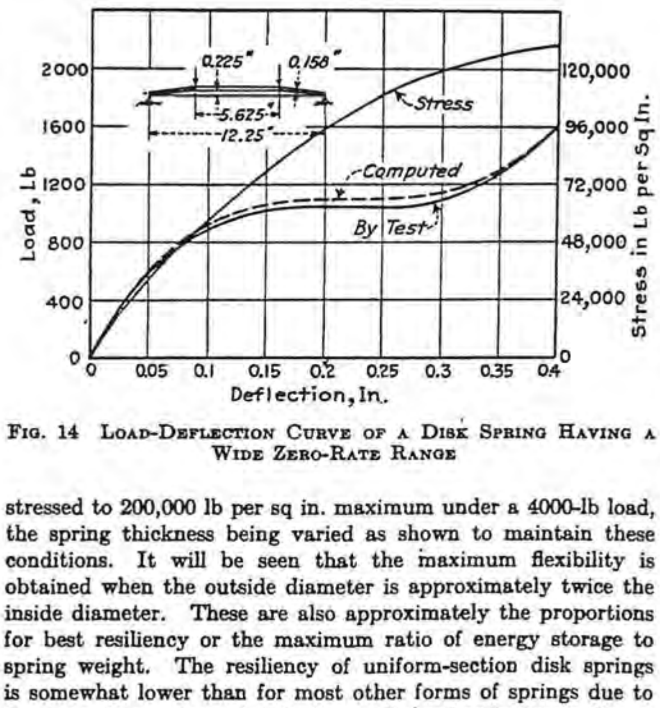

FIG. 14 LOAD-DEFLECTION CURVE OF A DISK SPRING HAVING A WIDE ZERO-RATE RANGE

stressed to 200,000 lb per sq in. maximum under a 4000-lb load, the spring thickness being varied as shown to maintain these conditions. It will be seen that the maximum flexibility is obtained when the outside diameter is approximately twice the inside diameter. These are also approximately the proportions for best resiliency or the maximum ratio of energy storage to spring weight. The resiliency of uniform-section disk springs is somewhat lower than for most other forms of springs due to the nonuniform stress distribution. Somewhat better resiliency can be obtained from disk springs with radially tapered sections as shown by Brecht and Wahl[3] but the added resiliency of such springs is obtained at the expense of increased cost. As a rough approximation, it may be said that the uniform-section disk spring has a resiliency one half that of a coil spring.

[3] "The Radially Tapered Disk Spring," by W. A. Brecht and A. M. Wahl, Trans. A.S.M.E., vol. 52, part 1, 1930, paper APM-57-4, pp. 45–55.

TANGENTIAL STRESSES IN ANNULAR-DISK SPRINGS

True stresses in the annular-disk spring are unknown. However, a background of experience is available which has established theoretical values which may be used for static loading. In dynamic applications, fatigue tests must be made.

Fig. 13 shows calculated stress-deflection curves for the springs, the characteristics of which are shown in Fig. 3. The solid lines in Fig. 13 are calculated for the upper edge of the inner circumference and the dashed lines for the lower edge of the inner circumference. Usually it is sufficient to calculate the stresses

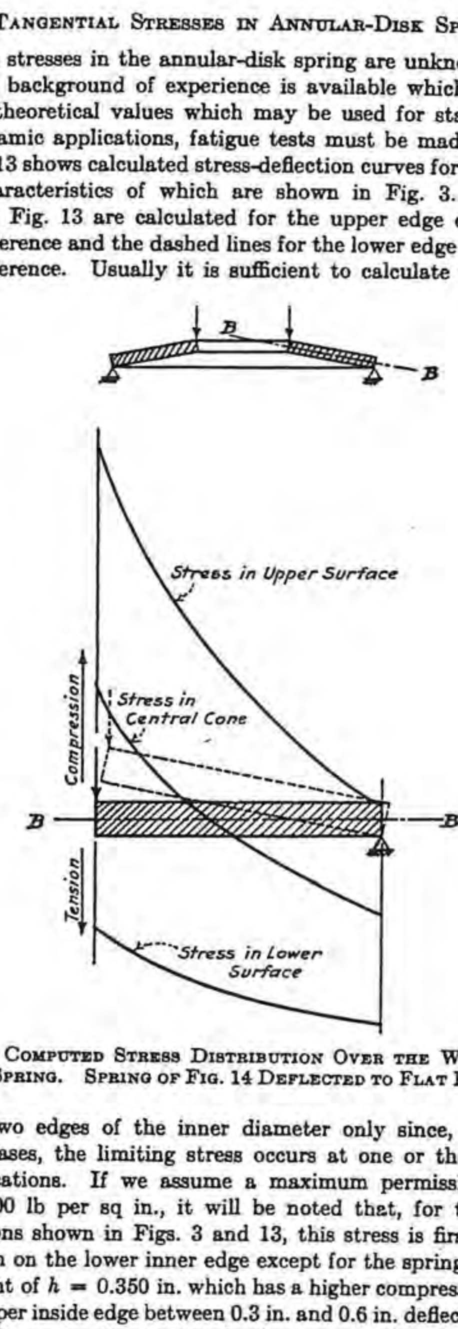

FIG. 15 COMPUTED STRESS DISTRIBUTION OVER THE WIDTH OF A DISK SPRING. SPRING OF FIG. 14 DEFLECTED TO FLAT POSITION

at the two edges of the inner diameter only since, except in special cases, the limiting stress occurs at one or the other of these locations. If we assume a maximum permissible stress of 200,000 lb per sq in., it will be noted that, for the spring proportions shown in Figs. 3 and 13, this stress is first reached in tension on the lower inner edge except for the spring having a free height of $h = 0.350$ in. which has a higher compressive stress at the upper inside edge between 0.3 in. and 0.6 in. deflection.

In addition to the load-deflection curve, Fig. 14 shows a calculated stress-deflection curve for a so-called zero-rate spring ($h = t\sqrt{2}$). These stress values are for the upper edge of the inside circumference.

In Fig. 15, the diagram shows in qualitative manner the stress distribution across the radial width of the disk spring shown in Fig. 14 when deflected to the flat position. Note from Fig. 15 that the upper surface is stressed in compression and that the maximum stress occurs at the edge of the inside diameter. The lower surface is stressed in tension but nowhere is the tension stress as great as the maximum compression stress on the upper surface. The stress in the central cone *B-B* is also shown, mainly to call attention to the fact that in springs of this type the central cone is not a neutral surface. The basis for the stress distribution as shown in Fig. 15 will be apparent from Fig. 17 and from Appendix 1 where the formulas are derived.

Fig. 16 shows the stress distribution across the thickness at the inner circumference for this same spring and for the same conditions as for Fig. 15. The stress at the upper and lower edges will, of course, be the same as for Fig. 15. The stress at intermediate points is represented by the line joining these extremes.

It must be noted from Fig. 15 that the stress distribution for other springs will be different from that shown and will also be different for other deflections of the same spring as may be seen from Fig. 13.

In the practical design of disk springs, it is found that they

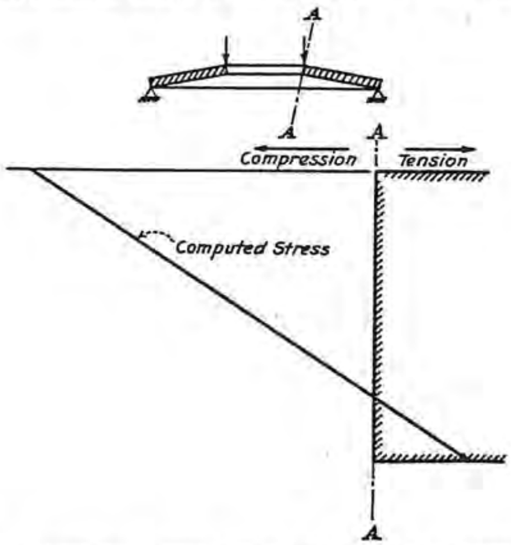

FIG. 16 COMPUTED STRESS OVER THE THICKNESS AT THE INNER CIRCUMFERENCE OF A DISK SPRING. SPRING OF FIG. 14 DEFLECTED TO FLAT POSITION

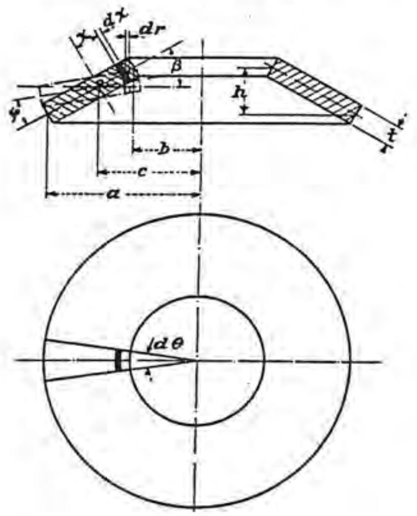

FIG. 17

will function satisfactorily under static loading when operating under computed stress as high as 200,000 to 220,000 lb per sq in. even though made from steel having a yield point of 120,000 lb per sq in. The apparent high stress capacity of disk springs may, in part, be due to shortcomings of the stress formula, due in turn to simplifying assumptions. These high values for computed stress are at the inner circumference. If it is assumed that the stress exceeds the yield point, there must be a redistribution of stress due to localized yielding. If, when loaded, the stress in the more highly stressed regions of the disk spring were redistributed as a result of yield, it follows that the unloaded spring will have residual stresses. This is substantiated by the fact that tests show a small initial loss in free cone height. The stress distribution, as shown by Figs. 15 and 16, suggests that important increase in fatigue life should result from careful rounding of the corners of disk springs. Limited test data support this view.

The safe stress of disk springs will, of course, vary with the type of service. There are not sufficient data now available to fix stress limits for dynamic applications. The fatigue tests which have been run indicate that, for a moderate stress range, a computed maximum stress of 180,000 lb per sq in. may be used. Only fatigue tests on actual springs can finally determine the true working limits.

Appendix 1—Derivation of Formulas for Disk Springs

The following nomenclature is used in deriving the formulas for disk springs:

- a = outer radius of disk
- b = inner radius of disk
- c = distance of neutral axis to center
- α = a/b = outer radius/inner radius
- β = initial cone angle of disk
- φ = change of cone angle due to load P
- P = axial load, uniformly applied around circumference
- h = free height of disk, measured as the elevation of the truncated cone formed by either the upper or lower surface
- δ = axial deflection of disk
- E = modulus of elasticity, taken as 30,000,000 lb per sq in.
- S = maximum stress
- r = spring rate
- V = strain energy
- ζ = resilience
- σ = Poisson's ratio, for steel $\sigma = 0.3$
- ϵ_1 = radial strain
- ϵ_2 = tangential strain
- κ_1 = change of radial curvature
- κ_2 = change of tangential curvature
- D = flexural rigidity

GENERAL CASE—INITIALLY CONED DISK SPRINGS

The method[4] used follows in general that used by S. Timoshenko[5] by assuming that the radial stresses are negligible and the cross section of the disk does not distort, but rather that it merely rotates about a neutral point O shown in Fig. 17.

(a) *Load and Deflection.* Consider a sector $d\theta$ of the disk in Fig. 17 and in it a strip dx at location x taking O as the origin. When the disk is deflected through an angle φ, this strip moves into its position, indicated by dashed lines. The ensuing tangential strain may be analyzed as the resultant of a radial displacement dr and a rotation φ. The first of these causes a uniform strain throughout the thickness of the disk if one neglects the small variation in distance to the center of the disk at various points of the section. The second results in a tangential bending strain which is zero in the neutral surface and maximum at the upper and lower surfaces. The tangential stresses produced by these two components of the strain cause a radial moment

[4] A similar method was used by W. A. Brecht and A. M. Wahl in developing equations for radially tapered disk springs. See Trans. A.S.M.E., vol. 52, part 1, 1930, paper, APM-52-4, p. 65.

[5] "Strength of Materials," by S. Timoshenko, D. Van Nostrand Company, New York, N. Y., 1934, vol. 2, p. 527.

about point O which resists the moment created by the external forces.

Calculating the tangential stress due to the radial displacement and first, we can write:

The length of section dx before deflection

$$l_1 = d\theta\,[c - x\cos\beta]$$

After deflection

$$l_2 = d\theta\,[c - x\cos(\beta - \varphi)]$$

The change in length

$$l_1 - l_2 = d\theta\,[-x\cos\beta\,(1 - \cos\varphi) + x\sin\beta\sin\varphi]$$

Substituting for small angles

$$\cos\beta = 1;\ \sin\beta = \beta;\ \sin\varphi = \varphi;\ 1 - \cos\varphi = 2\sin^2\varphi = \frac{\varphi^2}{2}$$

$$l_1 - l_2 = d\theta x\varphi\,(\beta - \varphi/2)$$

and the tangential strain will be approximately

$$\epsilon_2 = \frac{l_1 - l_2}{l_1} = \frac{x\,\varphi\,(\beta - \varphi/2)}{c - x}$$

The tangential stress[6]

$$S'_2 = \frac{E}{1 - \sigma^2}(\epsilon_2 + \sigma\epsilon_1)$$

Since it was assumed that the radial stresses are negligible, we can write

$$S'_2 = \frac{E\epsilon_2}{1 - \sigma^2} = \frac{Ex\varphi\,(\beta - \varphi/2)}{(1 - \sigma^2)\,(c - x)} \quad [1]$$

As a next step, we calculate the radial moment of the tangential forces in the section about point O.

$$dM_1' = S_2' t dx d\theta x \sin(\beta - \varphi)$$

Substituting $\sin(\beta - \varphi) = \beta - \varphi$ and also Equation [1]

$$dM_1' = \frac{Etd\theta\varphi\,(\beta - \varphi)\,(\beta - \varphi/2)\,x^2\,dx}{(1 - \sigma^2)\,(c - x)}$$

and integrating from $x = c - a$ to $x = c - b$, we get the internal moment of the sector about O.

$$M_1' = \frac{Etd\theta\varphi\,(\beta - \varphi)\,(\beta - \varphi/2)}{1 - \sigma^2}\left[\frac{1}{2}(a^2 - b^2) - 2c\,(a - b) + c^2\log\frac{a}{b}\right] \quad [2]$$

Calculating now the tangential stress due to the bending strain mentioned above, we can write the expression for the tangential bending moment per unit length.[7]

$$M_2 = D\,(\kappa_2 + \sigma\kappa_1)$$

This expression is positive as the change of curvature is positive in the case of a conical shell decreasing its height.

Bending in the radial section being neglected, we have for a section of length dx

$$dM_2 = D\kappa_2\,dx = \frac{Et^3}{12\,(1 - \sigma^2)}\kappa_2 dx$$

The tangential curvature of the unloaded disk is approximately

$$\frac{\sin\beta}{c - x}$$

and that of the deflected one

$$\frac{\sin(\beta - \varphi)}{c - x}$$

Hence the change of curvature

$$\kappa_2 = \frac{\sin\beta - \sin(\beta - \varphi)}{c - x}$$

Substituting $\sin\beta = \beta$

$$\sin(\beta - \varphi) = \beta - \varphi$$

$$\kappa_2 = \frac{\varphi}{c - x}$$

Hence the moment

$$dM_2 = \frac{E\,t^3\,\varphi\,dx}{12\,(1 - \sigma^2)\,(c - x)}$$

Then the tangential stress at the surfaces

$$S_2'' = \frac{6\,dM_2}{t^2 dx} = \frac{E\,\varphi}{(1 - \sigma^2)\,(c - x)}\frac{t}{2} \quad [3]$$

and at any point y distant from the neutral surface

$$S_2'' = \frac{E\,\varphi}{(1 - \sigma^2)\,(c - x)}\,y \quad [3a]$$

The radial component of the moments dM_2 in sector $d\theta$

$$dM_1 = 2\,dM_2\frac{d\theta}{2} = \frac{Et^3\varphi d\theta}{12\,(1 - \sigma^2)}\frac{dx}{(c - x)}$$

Integrating for the whole sector from $x = c - a$ to $x = c - b$

$$M_1'' = \frac{Et^3\varphi d\theta}{12\,(1 - \sigma^2)}\int_{c-a}^{c-b}\frac{dx}{c - x} = \frac{Et^3\varphi d\theta}{12\,(1 - \sigma^2)}\log\frac{a}{b} \quad [4]$$

Summing the radial moments (Equations [2] and [4]) the total radial moment

$$M_1 = M_1' + M_1'' = \frac{E\varphi d\theta}{1 - \sigma^2}\left\{\left[\frac{1}{2}(a^2 - b^2) - 2c\,(a - b) + c^2\log\frac{a}{b}\right](\beta - \varphi)\left(\beta - \frac{\varphi}{2}\right)t + \frac{t^3}{12}\log\frac{a}{b}\right\}$$

The value of c yet remains to be determined. This we get from the conditions of equilibrium on the sector, that the sum of all forces acting normal to the cross section must be equal to zero. Only stresses due to the radial displacement need be considered, however, since those due to bending have no resultant tangential force at any section x.

$$\int_{c-a}^{c-b} S_2' t dx = 0$$

or substituting S_2' from Equation [1]

$$\int_{c-a}^{c-b}\frac{xdx}{c - x} = 0$$

from which we get

$$c = \frac{a - b}{\log\frac{a}{b}} \quad [5]$$

[6] "A Treatise on the Mathematical Theory of Elasticity," third edition, by A. E. H. Love, Cambridge University Press, London, England, p. 533.

[7] Ibid., 1290, p. 533.

Substituting this into the moment equation

$$M_1 = \frac{E\varphi d\theta}{1-\sigma^2}\left\{\left[\frac{1}{2}(a^2-b^2) - 2\frac{(a-b)^2}{\log\frac{a}{b}} + \frac{(a-b)^2}{\log\frac{a}{b}}\right](\beta-\varphi)\left(\beta-\frac{\varphi}{2}\right)t + \frac{t^3}{12}\log\frac{a}{b}\right\}$$

The external moment on sector $d\theta$ equals

$$\frac{P\,(a-b)\,d\theta}{2\pi}$$

This must equal the internal moment, hence

$$P = \frac{2\,\pi M_1}{(a-b)\,d\theta}$$

Substituting the expression for M_1 and also

$$\beta = \frac{h}{a-b}$$

$$\varphi = \frac{\delta}{a-b}$$

$$\frac{a}{b} = \alpha$$

we get

$$P = \frac{E\,\delta}{(1-\sigma^2)a^2}\left\{\left[\frac{\alpha+1}{\alpha-1} - \frac{2}{\log\alpha}\right]\pi\left(\frac{\alpha}{\alpha-1}\right)^2 t\,(h-\delta)\left(h-\frac{\delta}{2}\right) + \frac{t^3}{6}\pi\log\alpha\left(\frac{\alpha}{\alpha-1}\right)^2\right\}$$

If we call $\left[\frac{\alpha+1}{\alpha-1} - \frac{2}{\log\alpha}\right]\pi\left(\frac{\alpha}{\alpha-1}\right)^2 = \frac{1}{M}$

and $\frac{\pi}{6}\log\alpha\left(\frac{\alpha}{\alpha-1}\right)^2 = \frac{1}{N}$ .... [6]

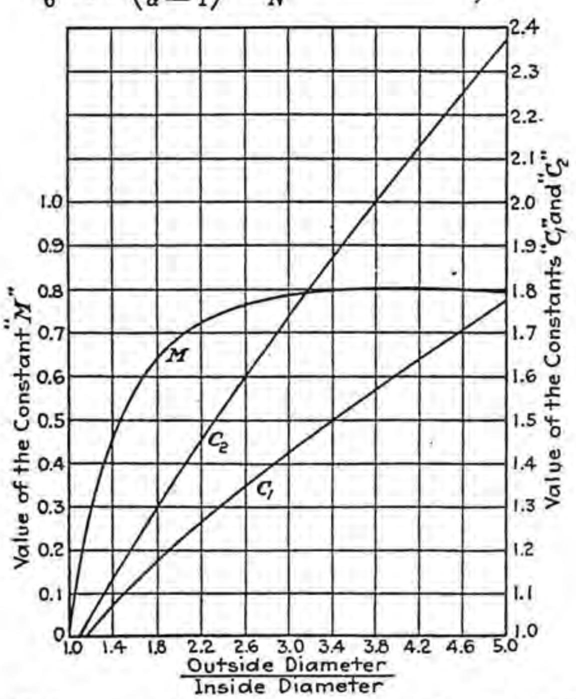

FIG. 18 LOAD AND STRESS CONSTANTS FOR CALCULATING DISK SPRINGS

$$P = \frac{E\,\delta}{(1-\sigma^2)\,a^2}\left[(h-\delta)\left(h-\frac{\delta}{2}\right)\frac{t}{M} + \frac{t^3}{N}\right]$$

Investigation of simultaneous values of M and N show them to be equal for practical purposes, so that

$$P = \frac{E\,\delta}{(1-\sigma^2)\,Ma^2}\left[(h-\delta)\left(h-\frac{\delta}{2}\right)t + t^3\right] \ldots\ldots [7]$$

The value of M for various ratios of $\frac{\text{outside diameter}}{\text{inside diameter}}$ are plotted in Fig. 18.

(*b*) *Maximum Stress.* The total tangential stress is the sum of its two components S_2' and S_2''. Adding Equations [1] and [3*a*]

$$S = S_2' + S_2'' = \frac{E\,\varphi}{(1-\sigma^2)\,(c-x)}\left[x\left(\beta-\frac{\varphi}{2}\right) + y\right] \ldots [8]$$

Its maximum value is reached at

$$y = \frac{t}{2} \text{ and } x = c - b$$

$$S = \frac{E\,\varphi}{(1-\sigma^2)b}\left[(c-b)\left(\beta-\frac{\varphi}{2}\right) + \frac{t}{2}\right]$$

or since $\beta = \frac{h}{a-b}$ and $\varphi = \frac{\delta}{a-b}$

$$S = \frac{E\,\delta}{(1-\sigma^2)\,(a-b)^2}\left[\left(\frac{c}{b}-1\right)\left(h-\frac{\delta}{2}\right) + \frac{t}{2}\,\frac{a-b}{b}\right]$$

Substituting $c = \frac{a-b}{\log\frac{a}{b}}$ and $\frac{a}{b} = \alpha$

$$S = \frac{E\delta}{(1-\sigma^2)\,a^2}\left[\left(\frac{\alpha-1}{\log\alpha}-1\right)\left(\frac{\alpha}{\alpha-1}\right)^2\left(h-\frac{\delta}{2}\right) + \frac{\alpha-1}{2}\left(\frac{\alpha}{\alpha-1}\right)^2 t\right]$$

Let us call

$$\left(\frac{\alpha-1}{\log\alpha}-1\right)\left(\frac{\alpha}{\alpha-1}\right)^2 M = \left(\frac{\alpha-1}{\log\alpha}-1\right)\frac{6.}{\pi\log\alpha} = C_1$$

and

$$\frac{\alpha-1}{2}\left(\frac{\alpha}{\alpha-1}\right)^2 M = \frac{\alpha-1}{2}\,\frac{6}{\pi\log\alpha} = \frac{3\,(\alpha-1)}{\pi\log\alpha} = C_2$$

where $M = N = \frac{6}{\pi\log\alpha}\left(\frac{\alpha-1}{\alpha}\right)^2$ from Equation [6]

Then

$$S = \frac{E\,\delta}{(1-\sigma^2)\,Ma^2}\left[C_1\left(h-\frac{\delta}{2}\right) + C_2\,t\right] \ldots\ldots [9]$$

Values of C_1 and C_2 are plotted against outside-diameter/inside-diameter ratio in Fig. 18.

If the maximum stress in the lower edge is sought, evidently $-t/2$ has to be substituted into 8 and the stress equation becomes

$$S = \frac{E\,\delta}{(1-\sigma^2)\,Ma^2}\left[C_1\left(h-\frac{\delta}{2}\right) - C_2 t\right] \ldots\ldots 10]$$

Both Equations [9] and [10] represent compressive stresses as long as the bracketed quantity is positive. Obviously, both can become negative at some deflection, in which case the stress is tension.

By equating Equations [9] and [10] it can be shown that the numerical value of the stress in the lower edge reaches that of the upper edge at a deflection $\delta = 2h$. It follows that at $\delta < 2h$ deflections, Equation [9] has to be used representing compressive stress in the upper edge, while at $\delta > 2h$ deflections, Equation [10] will give the greater value giving the tension stress in the lower edge of the spring.

Because in some cases the maximum working deflection is past the point where Equation [9] would give its greatest value, it is advisable in doubtful cases to plot the stress-deflection curve for both upper and lower inner edges of the spring.

As seen, for instance, on the last curve A in Fig. 13, the compressive stress in the upper inner edge exceeds the maximum permissible stress, say 200,000 lb per sq in. at a deflection of $\delta = 0.25$ in., whereas in the deflection range of 0.67 to 0.72 in. both Equations [9] and [10] would give lower values than 200,000 lb per sq in. If the maximum working deflection is between the two limits mentioned, the calculated stress value at that point would not indicate that the permissible stress limit has already been exceeded at a much smaller deflection.

(*c*) *Spring Rate.* Differentiating Equation [7] gives an expression for the spring rate at any point of the load-deflection curve. Designating spring rate by r, we have

$$r = \frac{Et}{(1-\sigma^2)\,Ma^2}\left(h^2 - 3\,\delta h + \frac{3}{2}\delta^2 + t^2\right) \ldots\ldots [11]$$

Equating the bracketed expression to zero, we can find the deflections at which the spring rate equals zero, i.e., where deflection may be increased without change in load.

These are

$$\delta = h + \sqrt{\frac{h^2}{3} - \frac{2\,t^2}{3}} \ldots\ldots [12]$$

When $\frac{h^2}{3} > \frac{2\,t^2}{3}$ or $h > t\sqrt{2}$ .... [13]

there will be two real values of δ corresponding to maximum and minimum values of P and the spring rate is negative between them, Fig. 6. If $h = t\sqrt{2}$, a spring is obtained which has only one point where the rate is zero and that at a deflection $\delta = h$, i.e., when the spring is flattened, Fig. 14. In cases where $h < t\sqrt{2}$, the spring rate is always positive.

If the condition for the so-called zero-rate spring

$$\delta = h = t\sqrt{2} \ldots\ldots [14]$$

is substituted into Equation [7], we get

$$P = \frac{\sqrt{2}\ E}{(1-\sigma^2)\,Ma^2}\,t^4 \ldots\ldots [15]$$

which is the load carried by such a spring at the point of zero spring rate. The corresponding stress from Equation [9]

$$S = \frac{t^2\sqrt{2}\,E}{(1-\sigma^2)\,Ma^2}\left(\frac{\sqrt{2}}{2}\,C_1 + C_2\right) \ldots\ldots [16]$$

There is a further significant relationship. Considering Equation [7], the load P may become zero if the bracketed quantity vanishes. Thus writing

$$(h-\delta)\left(h - \frac{\delta}{2}\right)t + t^3 = 0$$

and solving for δ

$$\delta = \frac{3}{2}h + \sqrt{\frac{h^2}{4} - 2\,t^2} \ldots\ldots [17]$$

If $\frac{h^2}{4} > 2\,t^2$ or $h > t\sqrt{8}$ .... [18]

then the load-deflection curve intersects the zero-load axis, that is, we have a buckling spring, one which would snap into a new position once deflected beyond a certain point. If $\sqrt{2} < \frac{h}{t} < \sqrt{8}$ we shall have a spring with negative spring rate, but one which will not buckle.

(*d*) *Initially Flat Disk Springs.* The equations for initially flat springs can be derived by substituting $h = 0$ into those developed for initially coned springs. Thus we have for the load

$$P = \frac{E\,\delta}{(1-\sigma^2)\,Ma^2}\left(\frac{\delta^2 t}{2} + t^3\right) \ldots\ldots [19]$$

for the maximum stress

$$S = \frac{E\,\delta}{(1-\sigma^2)\,Ma^2}\left(C_1\frac{\delta}{2} + C_2 t\right) \ldots\ldots [20]$$

which in this case is always tension in the lower inner edge of the spring.

From Equation [19], the deflection can be expressed as

$$\delta = \frac{PMa^2}{Et^3}\ \frac{1-\sigma^2}{1 + 0.5\,\dfrac{\delta^2}{t^2}}$$

The rate becomes

$$r = \frac{Et}{(1-\sigma^2)\,Ma^2}\left(\frac{3}{2}\delta^2 + t^2\right) \ldots\ldots [21]$$

(*e*) *Resilience of Disk Springs.* The strain energy for δ deflection is

$$V = \int_0^\delta P d\delta \ldots\ldots [22]$$

If we substitute into this P from Equation [7] and integrate, we have

$$V = \frac{E\delta^2}{2\,(1-\sigma^2)\,Ma^2}\left[t\left(h - \frac{\delta}{2}\right)^2 + t^2\right] \ldots\ldots [23]$$

The volume of the spring may be written as $(a^2 - b^2)\,\pi\,t = a^2\,\frac{\alpha^2 - 1}{\alpha^2}\,\pi\,t$ so the resilience or the strain energy per unit volume

$$\zeta = \frac{E\,\delta^2}{2\,\pi\,(1-\sigma^2)\,Ma^4}\left(\frac{\alpha^2}{\alpha^2 - 1}\right)\left[\left(h - \frac{\delta}{2}\right)^2 + t^2\right] \ldots [24]$$

If the resilience for a given maximum permissible stress is sought, the corresponding value of δ may be computed from Equations [9] or [10].

Appendix 2—Calculation of Disk Springs

Initially Coned Disk Springs

When calculating disk springs with large deflections, the formulas given in Appendix 1 do not lend themselves to an easy evaluation of the dimensions. In most cases of practical calculation, the load, outside and inside diameters of the spring, the maximum permissible stress, and a general type of load-deflection curve are given which leaves the deflection, thickness, and free height to be calculated.

Deflection Equation [7] does not lend itself to ready solution even when the thickness and free cone height are known as the equation is cubic; therefore, it is recommended that the following method of approach be employed.

As, in most cases, a low spring rate is desirable, assume that the desired load is carried by the spring when it has its lowest rate, that is, when $\delta = h$, or when the spring is flattened. Equation [7] then simplifies to

$$P = \frac{Eht^3}{(1 - \sigma^2)\, Ma^2} \qquad [25]$$

The ratio of h/t has to be assumed now, bearing in mind that as shown in Appendix 1 if $(h/t) < \sqrt{2}$ we get a spring with positive, though variable, rate as shown in Fig. 5. If $(h/t) = \sqrt{2}$ the spring will have a point of zero rate as shown in Fig. 14. If $(h/t) > \sqrt{2}$ it will have a range of negative rate as shown in Fig. 6. Finally, if $(h/t) > \sqrt{8}$ it becomes a buckling spring. The thickness and free height can now be computed and since $\delta = h$, the deflection is also known. These values have to be substituted now in Equations [9] or [10] as explained in Appendix 1, and the maximum stress computed. If that is too high, a lower value for h/t has to be chosen meaning, of course, an increased spring rate in the flattened position. A few trial calculations are usually sufficient.

If a low spring rate has to be maintained at the specified loading, four courses are open: (1) Change the outside-diameter-inside-diameter ratio so that it will lie between 1.6 and 2.4; (2) increase the outside diameter; (3) decrease the load on each spring by using two or more springs in parallel; and (4) increase deflection by using several springs in series.

Example. Calculate a spring with the following data: Load $P = 1000$ lb, maximum permissible stress $S = 200{,}000$ lb per sq in., outside diameter $2a = 6$ in., inside diameter $2b = 3$ in. From Fig. 18, $M = 0.69$, $C_1 = 1.225$, and $C_2 = 1.38$; also $1 - \sigma^2 = 0.91$ if $\sigma = 0.3$.

Assuming that a zero rate is desirable when carrying the specified load, we take $h/t = \sqrt{2}$, and so from Equation [25].

$$t = \sqrt[4]{\frac{P(1 - \sigma^2)\, Ma^2}{\sqrt{2}E}} = \sqrt[4]{\frac{1000 \times 0.69 \times 9 \times 0.91}{1.414 \times 30 \times 10^6}} = 0.107 \text{ in.}$$

$$h = 0.151 \text{ in.} = \delta$$

The maximum stress from Equation [9]

$$S = \frac{E\delta}{(1 - \sigma^2)\, Ma^2}\left(C_1 \frac{h}{2} + C_2 t\right) = \frac{30 \times 10^6 \times 0.151}{0.69 \times 9 \times 0.91}\,(1.225 \times 0.076 + 1.38 \times 0.107) = 198{,}000 \text{ lb per sq in.}$$

Now the load-deflection curve can be plotted assuming various values of δ and calculating P from Equation [7].

Initially Flat Springs

The procedure used for initially coned disk springs cannot be used when dealing with initially flat springs. In this case, the following method is applicable. Equations [19] and [20] for the calculation of load and minimum stress are

$$P = \frac{E\,\delta}{(1 - \sigma^2)\, Ma^2}\left(\frac{\delta^2 t}{2} + t^3\right)$$

$$S = \frac{E\delta}{(1 - \sigma^2)\, Ma^2}\left(C_1 \frac{\delta}{2} + C_2 t\right)$$

Dividing the two equations and simplifying, we have

$$\delta^2 \frac{S}{P} = \frac{\dfrac{C_1}{2} + C_2 \dfrac{t}{\delta}}{\dfrac{1}{2}\dfrac{t}{\delta} + \left(\dfrac{t}{\delta}\right)^3} = A^2 \text{ or}$$

$$A = \delta \sqrt{\frac{S}{P}} \qquad [26]$$

where A is a function of the ratio t/δ and α.

Again, from Equation [19], we get

$$P = \frac{E\delta^4}{(1 - \sigma^2)\, Ma^2}\left[\frac{1}{2}\frac{t}{\delta} + \left(\frac{t}{\delta}\right)^3\right] \text{ or}$$

$$P = \frac{E\delta^4}{(1 - \sigma^2)\, Ma^2} B^2 \qquad [27]$$

where

$$B = \sqrt{\frac{1}{2}\frac{t}{\delta} + \left(\frac{t}{\delta}\right)^3}$$

The various values of A and B are plotted on a logarithmic scale against t/δ in Fig. 19. The procedure is that of trial and error.

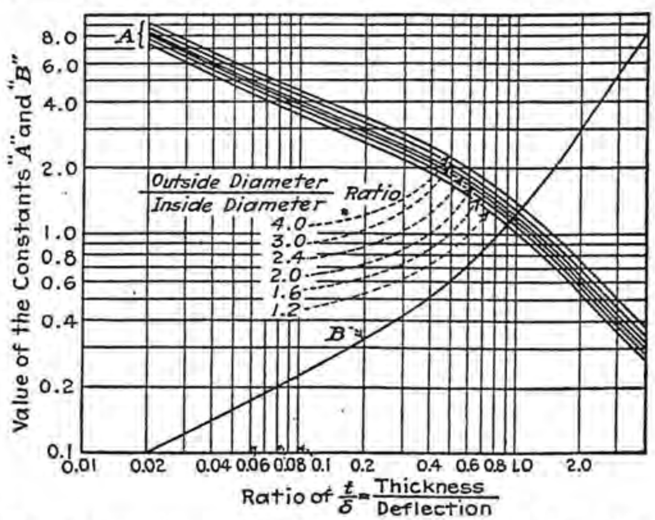

Fig. 19 Constants for Calculating Initially Flat Disk Springs

Assuming that the load, the maximum permissible stress, and outside and inside diameters are given, we proceed by taking first an arbitrary value for the deflection δ and calculate the value A from Equation [26]. The value of B on the same vertical scale can then be read off the curve given in Fig. 19, and substituted in Equation [27] which then gives a value for the load. If this is higher than the desired load, a smaller deflection has to be assumed and the process repeated. Vice versa, too small a load calls for an increase of δ. After several trials, one can usually obtain the desired load, then read from Fig. 19 the value t/δ corresponding to the value of B used last. This gives all the relevant information.

Example. Find thickness and deflection for a spring with the following data: Load $P = 2000$ lb, maximum permissible stress $S = 200{,}000$ lb per sq in., outside diameter $2a = 6$ in., inside diameter $2b = 3$ in.

Then from Fig. 18 the load constant $M = 0.69$.

Assume $\delta = 0.4$ in., then from Equation [26]

$$A = 0.4 \sqrt{\frac{200{,}000}{2000}} = 0.4 \times 10 = 4$$

From Fig. 19, $B = 0.21$.
From Equation [23]

$$P = \frac{30 \times 10^6 \times 0.4^4 \times 0.21^2}{0.91 \times 0.69 \times 3^2} = 6000 \text{ lb}$$

As a second trial, take now $\delta = 0.2$. Then $A = 2$. $B = 0.55$ and

$$P = \frac{30 \times 10^6 \times 0.2^4 \times 0.55^2}{0.91 \times 0.69 \times 3^2} = 2570 \text{ lb}$$

Again, take $\delta = 0.16$. Then $A = 1.6$, $B = 0.76$

and

$$P = \frac{30 \times 10^6 \times 0.16^4 \times 0.76^2}{0.91 \times 0.69 \times 3^2} = 2010 \text{ lb}$$

If this is considered sufficiently close, we get $t/\delta = 0.64$ from Fig. 19 or $t = 0.64 \times 0.16 = 0.102$ in.