# THEORY FOR THE CALCULATION OF THE TANGENTIAL RESIDUAL STRESS DISTRIBUTION IN CURVED BEAMS

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## ABSTRACT

A method is developed for the determination of tangential residual stress distribution in curved beams of rectangular cross-section. The determination of such distributions is based upon the fact that when a residually stressed and thin layer is removed from the periphery of the beam, the radius of curvature of the beam changes. By knowing this change, it is possible to calculate the stress in the layer removed.

The method developed may be handled on a desk calculator, although it may be extremely laborious. The method is more easily handled using machine computation facilities.

## INTRODUCTION

This report presents a theory for the calculation of the tangential residual stress distribution in curved beams of constant rectangular cross-section. The theory is based on the fact that if such a beam is residually stressed and a thin layer is removed from its periphery the radius of curvature of the beam will change. By knowing this change, it is possible to calculate the stress that was in the layer removed. The assumption is made that the residual stresses in the radial and axial directions are negligible.

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The equation developed in this report for the residual stress in a curved beam is complicated and would be extremely laborious to evaluate with a desk calculator. However, the equation can easily be evaluated by a digital computer. If the digital computer is not available, the theory used for straight beams can be used on curved beams without a prohibitive error. For curved beams in which the ratio of the radius of curvature of the centroid of the beam to the radial thickness is 10:1, the theory used for straight beams may be applied with an error of less than 4 percent. For a ratio of 5:1, the error is less than 10 percent[1]*. However, for smaller ratios and for greater accuracy, it is necessary to use curved beam theory. The theory presented here is not exact, but the results obtained using it are in very good agreement with those obtained using the exact solution, which is considerably more complicated. The error is less than one-half percent for a radius to thickness ratio of 4:1 [2].

## THEORY

First it is necessary to understand what happens to a residually stressed curved beam when a thin layer is removed. Consider the beam of unit width shown in Fig. 1.

If in the infinitesimal thickness $dz$ there is a stress $s$, the removal of the layer $dz$ will remove a force $s \cdot dz$. Now this force at the surface of the beam can be resolved into a

* Superiors in brackets pertain to references listed at the end of the paper.

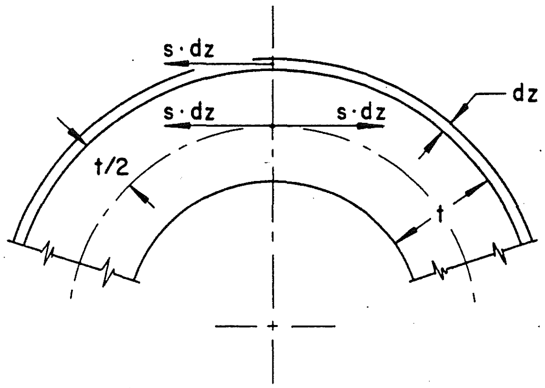


FIG. 1.

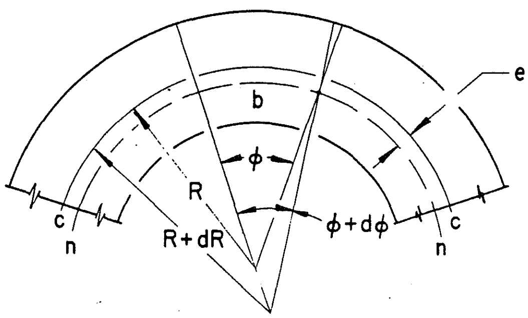


FIG. 2.

pure bending moment and a force that produces a direct stress uniformly distributed over the cross-section. For the latter the force must be applied at the centroid of the cross-section [3]. If this is done, the moment arm of the pure bending moment is $t/2$ as indicated in the figure, and its magnitude is $s \cdot dz \cdot t/2$. The magnitude of the direct stress is $s \cdot dz/t$.

Now consider the beam shown in Fig. 2 as the radius of curvature of the centroid changes from $R$ to $R+dR$ due to pure bending. The neutral axis is denoted by $n-n$ and the centroid by $c-c$. The angle $\phi$ subtends an arc length $b$ on the neutral axis as does the angle $\phi + d\phi$ and $e$ is the radial distance between the neutral axis and the centroid.

Now consider the ratio $d\phi/\phi$. This ratio appears in many of the equations of curved beams. To measure either quantity, however, would be difficult. But $d\phi/\phi$ may be expressed in terms of the radius of curvature of the centroid of the beam which is easily determined.

$$\frac{d\phi}{\phi} = \frac{(\phi + d\phi) - \phi}{\phi} = \frac{\dfrac{b}{R+dr-e} - \dfrac{b}{R-e}}{\dfrac{b}{R-e}}$$

$$= \frac{R-e-R-dR+e}{(R-e)(R+dR-e)} \cdot \frac{R-e}{1} = -\frac{dR}{R-e}, \qquad (1)$$

if $dR$ is very much less than $R$.

Finally consider the curved beam shown in Fig. 3. Let the tangential residual stress distribution in the beam be represented by $S(z,x)$ at any time during the dissection.

The variable $z$ refers to the thickness of metal removed from the outside surface, and the variable $x$ refers to the location of the stress below the original outside surface. At any time during the metal removal $x$ lies between $z$ and $t$, or is equal to $z$ or $t$. The initial stress distribution in the beam before any metal has been removed is $S(0,x)$, and the initial stress in the outside surface $S(0,0)$. The stress in the new outside surface after a thickness $z$ has been removed is $S(z,z)$. The radius of curvature of the centroid of the beam is a function of the amount of metal that has been removed. Let this function be represented by $R(z)$.

Now suppose that a thickness $z$ has been removed from the outside of the curved beam, Fig. 3, and an additional elemental thickness $dz$ is being removed. The bending moment resulting from the removal of the layer $dz$ is given by

$$M = -S(z,z)\,dz\,\frac{t-z}{2}. \qquad (2)$$

The bar is assumed to be of unit width. $M$ is defined to be positive if the curvature of the beam decreases. But the bending moment is also given by [4]

$$M = -AeE\,\frac{d\phi}{\phi}, \qquad (3)$$

where $A$ is the area of the cross-section, $e$ is the distance from the centroid to the neutral axis, and $E$ is Young's modulus. The minus sign is needed if $M$ is defined as above.

Substituting (1) in (3), equating the right hand sides of (2) and (3), and rearranging terms gives

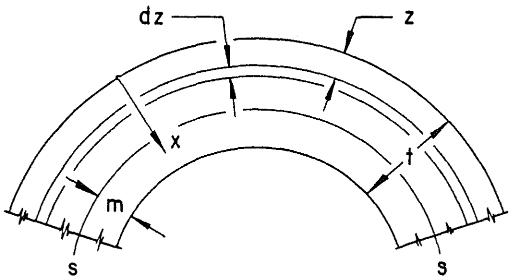


FIG. 3.

$$S(z,z) = -\frac{2AeE}{(t-z)\,dz}\,\frac{dR(z)}{R(z)-e}. \tag{4}$$

But $A = (t-z)$, and $dR(z)/dz = R'(z)$, so

$$S(z,z) = -\frac{2EeR'(z)}{R(z)-e}. \tag{5}$$

The cut $dz$ produces a change in stress at the section $s-s$, which is $m$ units outside the inside diameter, due to a change in direct stress and a change in bending stress. The change in direct stress is

$$dS_d = \frac{S(z,z)\,dz}{t-z}. \tag{6}$$

Substituting (5) in (6), we get

$$dS_d = -\frac{2EeR'(z)\,dz}{[R(z)-e](t-z)}. \tag{7}$$

The change in bending stress at section $s-s$ is [4]

$$dS_b = \frac{My}{Ae(r-y)}, \tag{8}$$

where $M$ is given by (2), $y$ is the distance from the neutral axis to section $s-s$ and is taken as positive toward the center of curvature, $A$ the area of the cross-section, $e$ the distance from the neutral axis to the centroid, and $r$ the radius of curvature of the neutral axis.

$$\begin{aligned} M &= -S(z,z)\,dz\,\frac{t-z}{2} = \frac{EeR'(z)\,dz\,(t-z)}{R(z)-e}, \\ y &= \frac{t-z}{2} - e - m, \\ A &= t-z, \qquad r = R(z) - e. \end{aligned} \tag{9}$$

Substituting Eqs. (9) into (8) gives

$$dS_b = \frac{ER'(z)[(t-z) - 2e - 2m]\,dz}{[2R(z) - t + z + 2m][R(z) - e]}. \tag{10}$$

The total change in stress at section $s-s$ due to the removal of the layer $dz$ is, adding (7) and (10),

$$\begin{aligned} dS = dS_d + dS_b &= \frac{-2EeR'(z)\,dz}{[R(z)-e][t-z]} \\ &+ \frac{ER'(z)[(t-z) - 2e - 2m]\,dz}{[2R(z) - t + z + 2m][R(z)-e]}. \end{aligned} \tag{11}$$

Now suppose that a layer $\bar{z} = t - m$ has been removed from the beam. The change in stress at section $s-s$ will be the sum of all the changes from each elemental thickness $dz$ removed. Therefore,

$$\begin{aligned} \int_0^{\bar{z}} dS = S(\bar{z},\bar{z}) - S(0,\bar{z}) &= -\int_0^{\bar{z}} \frac{2EeR'(z)\,dz}{[R(z)-e][t-z]} \\ &+ \int_0^{\bar{z}} \frac{ER'(z)[(t-z) - 2e - 2m]\,dz}{[2R(z) - (t-z) + 2m][R(z)-e]}. \end{aligned} \tag{12}$$

However, $S(z,z)$ is known from (5), and $m = t - \bar{z}$. Making these substitutions in (12) and rearranging terms gives

$$\begin{aligned} S(0,z) = &-\frac{2EeR'(\bar{z})}{R(\bar{z})-e} + 2E\int_0^{\bar{z}} \frac{eR'(z)\,dz}{[R(z)-e][t-z]} \\ &+ 2E(t-z)\int_0^{\bar{z}} \frac{R'(z)\,dz}{[R(z)-e][2R(z)+t+z-2\bar{z}]} \\ &- E\int_0^{\bar{z}} \frac{R'(z)[(t-z)-2e]\,dz}{[R(z)-e][2R(z)+t+z-2\bar{z}]}. \end{aligned} \tag{13}$$

Now as a first approximation [5],

$$e = \frac{(t-z)^2}{12R(z)}. \tag{14}$$

Making this substitution in (13) gives

$$\begin{aligned} S(0,\bar{z}) = &-\frac{2E(t-\bar{z})^2R'(z)}{12[R(\bar{z})]^2 - (t-\bar{z})^2} + 2E\int_0^{\bar{z}} \frac{(t-z)R'(z)\,dz}{12[R(z)]^2 - (t-z)^2} \\ &+ 2E(t-\bar{z})\int_0^{\bar{z}} \frac{12R(z)R'(z)\,dz}{\{12[R(z)]^2 - (t-z)^2\}[2R(z)+t+z-2\bar{z}]} \\ &- E\int_0^{\bar{z}} \frac{[12(t-z)R(z) - 2(t-z)^2]R'(z)\,dz}{\{12[R(z)]^2 - (t-z)^2\}[2R(z)+t+z-2\bar{z}]}. \end{aligned} \tag{15}$$

## CONCLUSION

It may finally be seen that Eq. (15) gives the stress at the depth $x = \bar{z}$ before any metal has been removed. Therefore, the equation gives the original tangential residual stress distribution in a curved beam, if the radius of curvature of the centroid of the beam is known as a function of the thickness of metal removed. Unfortunately, this will not be an analytic function so numerical methods are used to evaluate the equation. In order to determine $R(z)$, the radius of curvature of the centroid of the beam, $D(z)$, the inside diameter of the beam, is measured. Then $R(z)$ for any value is given by

$$R(z) = \frac{D(z)}{2} + \frac{t - z}{2}, \tag{16}$$

where $(t - z)$ is the radial thickness of the beam. By inspecting the equation for $R(z)$ it may be seen that $R(z)$ will change if $D(z)$ and/or $(t - z)$ change. The first term is due to bending and the second is due to a change in thickness. The only change in $R(z)$ that can be considered here is the change due to bending, so when the derivative of $R(z)$ is taken, $(t - z)$ must be considered a constant. Then $R'(z)$ for any value of $z$ is given by

$$R'(z) = \frac{D'(z)}{2}. \tag{17}$$

## INSIDE DIAMETER MEASUREMENT

It is essential that the inside diameter, $D(z)$, of the curved beam be measured with care, particularly when the derivative of $D(z)$ is small. At the Research Staff, the inside diameter is measured by means of a commercially available remote printing comparator having a minimum division reading of 0.00005 inch.

For example, if a curved beam having a nominal inside diameter of 0.5 inch and a wall thickness of 0.05 inch experiences a diametral change of 0.00005 inch when a layer 0.001 inch thick is removed, a stress change of about 4,000 psi is indicated.

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[3] Timoshenko, S., "Strength of Materials", p. 69.
[4] Ibid, p. 67.
[5] Ibid, p. 70.