Modeling of Shot Peening Residual Stresses with a Generalized J2 Plasticity Theory

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Abstract
An analytical residual stress model for a shot peened surface has been developed based on the generalization of J2 theory. Traditional elastoplasticity theory is utilized to develop a residual stress model dependent on both the second and third deviatoric stress invariants. A flow rule is obtained that is a function of the plastic hardening modulus. This allows us to combine the simplified behavior of the uniaxial stress strain curve with a generalized Von Mises yield function. We utilize this flow rule to calculate the elastic plastic stress deviators from which the residual stresses are found instead of relying on Iliushin's [1] results. However, for certain specialized cases our approach yields equivalent results.

Keywords Shot peening, residual stress, generalized material, incremental plasticity theory

Introduction
The compressive residual stress from shot peening is responsible for an increase in fatigue strength. Therefore, developing a theoretical model of the residual stresses not only helps make predictions from the peening parameters but also provides a better understanding of the physics. There exist only a handful of modeling formulations to predict the compressive residual stress [2 – 5] because of the complexities involved in the process. In the shot peening process a triaxial state of stress is induced during the impact. When a finite structure is peened the complex stress state makes prediction of yielding difficult. It is well known that for complex loading cases neither the Von Mises nor Tresca yield criteria are capable of predicting precisely when yielding occurs [6]. Drucker [7] formulated several alternate yield criteria for materials undergoing combined loading to allow for more flexibility when predicting experimental behavior. The yield criteria developed by Drucker is a modification of the Von Mises yield criteria and incorporates the third invariant of the deviatoric stress. By introducing the third invariant (J3) of the stress deviator, an analytical model is developed by assuming a more general isotropic material. The initial results from this model provided good agreement between analytical and published experimental results.

Theoretical Development
We begin by outlining the fundamental elasticity equations that describe the impact of a single spherical indenter which are developed from the theory of Hertzian contact [8]. The basic quantities developed from Hertzian contact include the elastic indentation radius and maximum normal elastic pressure

$$a_e = R\left(\frac{5\pi\rho\kappa V^2}{2E_0}\right)^{\frac{1}{5}} \quad (1)$$

$$p = \frac{1}{\pi}\left(\frac{5}{2}\pi\kappa\rho V^2 E_0^4\right)^{\frac{1}{5}} \quad (2)$$

where $\frac{1}{E_0} = \frac{1-\nu^2}{E} + \frac{1-\nu_s^2}{E_s}$ and v is the shot velocity, R is shot radius, ν is the target Poisson ratio, ν_s is the shot Poisson ratio, ρ is the shot density and κ is an efficiency coefficient used to take into account thermal dissipation and elastic rebound.

Table 1 show all the fundamental elastic quantities derived from the Hertzian stress field for a position directly below the shot. The deviatoric and mean stresses are given as s and σ_m respectively.

Table 1. Elasticity quantities for a spherical indenter impacting a semi infinite surface.

Stress Quantities	Strain Quantities
$\sigma_{11}^e = p(1+\nu)\left[\frac{x_3}{a_e}\tan^{-1}\left(\frac{x_3}{a_e}\right) - 1\right] + p\frac{a_e^2}{2(a_e^2 + x_3^2)}$	$\epsilon_{11}^e = \frac{1}{E}[\sigma_{11}^e - \nu(\sigma_{22}^e + \sigma_{33}^e)]$
$\sigma_{22}^e = p(1+\nu)\left[\frac{x_3}{a_e}\tan^{-1}\left(\frac{x_3}{a_e}\right) - 1\right] + p\frac{a_e^2}{2(a_e^2 + x_3^2)}$	$\epsilon_{22}^e = \frac{1}{E}[\sigma_{22}^e - \nu(\sigma_{11}^e + \sigma_{33}^e)]$
$\sigma_{33}^e = -p\left[\left(\frac{x_3}{a_e}\right)^{-1} + 1\right]$	$\epsilon_{33}^e = \frac{1}{E}[\sigma_{33}^e - \nu(\sigma_{22}^e + \sigma_{11}^e)]$
$s_{11}^e = \sigma_{11}^e - \sigma_m^e = \frac{1}{3}\sigma_i^e$	$e_{11}^e = \epsilon_{11}^e - \epsilon_m^e = \frac{1}{3}(1+\nu)\epsilon_i^e$
$s_{22}^e = \sigma_{22}^e - \sigma_m^e = \frac{1}{3}\sigma_i^e$	$e_{22}^e = \epsilon_{22}^e - \epsilon_m^e = \frac{1}{3}(1+\nu)\epsilon_i^e$
$s_{33}^e = \sigma_{33}^e - \sigma_m^e = -2s_{11}^e = -\frac{2}{3}\sigma_i^e$	$e_{33}^e = \epsilon_{33}^e - \epsilon_m^e = -2s_{11}^e = -\frac{2}{3}(1+\nu)\epsilon_i^e$
$\sigma_m^e = \frac{1}{3}(\sigma_{11}^e + \sigma_{22}^e + \sigma_{33}^e)$	$\epsilon_m^e = \frac{1}{3}(\epsilon_{11}^e + \epsilon_{22}^e + \epsilon_{33}^e)$

For simple J2 theory the effective stress is defined as

$$\sigma_i^e = \sqrt{3J_2} = \frac{1}{\sqrt{2}}[(\sigma_{11}^e - \sigma_{22}^e)^2 + (\sigma_{11}^e - \sigma_{33}^e)^2 + (\sigma_{22}^e - \sigma_{33}^e)^2]^{\frac{1}{2}} \tag{3}$$

However, for our analysis we introduce an alternate effective stress that depends on both invariants J2 and J3

$$\sigma_i^e = \sqrt{3J_2}\left(1 - c\frac{J_3^2}{J_2^3}\right)^{\frac{1}{2}} \tag{4}$$

The parameter c is typically restricted to the range of values $-3.375 \leq c \leq 2.25$ [6]. Note for $c = 0$ we have our effective stress for simple J2 theory. In Eq. (4) J2 and J3 can be defined in terms of the principal stresses

$$J_2 = \frac{1}{6}[(\sigma_{11}^e - \sigma_{22}^e)^2 + (\sigma_{11}^e - \sigma_{33}^e)^2 + (\sigma_{22}^e - \sigma_{33}^e)^2]$$

$$J_3 = \begin{vmatrix} \sigma_{11}^e - \sigma_m^e & 0 & 0 \\ 0 & \sigma_{22}^e - \sigma_m^e & 0 \\ 0 & 0 & \sigma_{33}^e - \sigma_m^e \end{vmatrix} \tag{5}$$

Also from Eq. (4) the effective strain is

$$\epsilon_i^e = \frac{\sigma_i^e}{E} \tag{6}$$

Now that we have defined the fundamental elasticity quantities we must develop the required plasticity equations. The goal is to relate our known, measureable elasticity expressions to the equations of plasticity. Our first step is to define the effective plastic strain

$$\epsilon_i^p = \begin{cases} \epsilon_i^e & for\ \epsilon_i^e \leq \epsilon_s \\ \epsilon_s + \alpha(\epsilon_i^e - \epsilon_s) & for\ \epsilon_i^e > \epsilon_s \end{cases} \tag{7}$$

Li [5] defines α as the ratio of the plastic indentation, a_p, and elastic indentation, a_e. ϵ_s is the yield strain. The elastic indentation can be calculated from Eq. (1). Miao [9] calculates the plastic indentation from

$$a_p = R\left[\frac{8\rho(V\sin\theta)^2}{9\sigma_s}\right]^{\frac{1}{4}} \tag{8}$$

σ_s is the yield strength. By assuming a multi-linear relationship between the elasto-plastic stress and strains allows us to write them as

$$\sigma_i^p = \begin{cases} \sigma_i^e & for\ \epsilon_i^p \leq \epsilon_s \\ \sigma_s + k(\epsilon_i^p - \epsilon_s) & for\ \epsilon_s \leq \epsilon_i^p < \epsilon_b \\ \sigma_b & for\ \epsilon_i^p \geq \epsilon_b \end{cases} \tag{9}$$

where σ_b is the tensile strength and k is the hardening coefficient. A great simplification can be made by assuming the plastic strain deviators take on a similar form as the elastic strain deviators. Because of axisymmetric loading and geometric considerations the elastic plastic strain deviators can be calculated from [5]

$$\begin{aligned} e_{11}^p &= e_{22}^p = \frac{1}{3}(1+\nu)\epsilon_i^p \\ e_{33}^p &= -\frac{2}{3}(1+\nu)\epsilon_i^p \end{aligned} \tag{10}$$

We assume that ν is .5 because of plastic flow. The next goal is to derive the elastic plastic stress deviator, s_{ij}^p. Li utilizes Iliushin's elastic plastic theory to obtain

$$\begin{aligned} s_{11}^p &= s_{22}^p = \frac{1}{3}\sigma_i^p \\ s_{33}^p &= -\frac{2}{3}\sigma_i^p \end{aligned} \tag{11}$$

An alternate approach is applied here. Drucker [7] expressed the plasticity flow rule as

$$de_{ij}^p = d\lambda\frac{\partial f}{\partial s_{ij}} = \bar{G}\partial f\frac{\partial f}{\partial s_{ij}} \tag{12}$$

f is our yield function, $d\lambda$ is a positive scalar and $\bar{G}$ is a scalar function. We can solve for $\bar{G}$ with simple algebra by squaring both sides of Eq. (12) and noting $\left(de_{ij}^p de_{ij}^p\right)^{\frac{1}{2}} = \sqrt{\frac{2}{3}}d\epsilon_i^p$. ∂f can be written in terms of the effective plastic stress $\partial f = \frac{2}{3}\sigma_i^p d\sigma_i^p$ therefore

$$\bar{G} = \frac{\sqrt{\frac{3}{2}}}{\frac{2}{3}\sigma_i^p H_p \left(\frac{\partial f}{\partial s_{ij}}\frac{\partial f}{\partial s_{ij}}\right)^{\frac{1}{2}}} \tag{13}$$

With $H_p = \frac{d\sigma_i^p}{d\epsilon_i^p}$ as the plastic hardening modulus.

The yield function $f = J_2\left(1 - c\frac{J_3^2}{J_2^3}\right)^{\alpha} - K^2$ has been proposed (see Plasticity Theory, Lubliner) with c satisfying the same range as Eq. (4). α is taken either as 1 or 1/3, 1 is used here and K is the yield stress in simple shear. After performing the appropriate calculations $\bar{G}$ has the form

$$\bar{G} = \frac{\sqrt{\frac{3}{2}}}{\frac{2}{3}\sigma_i^p H_p \left(\left(\frac{\partial f}{\partial J_2}\right)^2 2J_2 + \left(\frac{\partial f}{\partial J_3}\right)^2 \frac{2}{3}J_2^2 + 6\frac{\partial f}{\partial J_3}\frac{\partial f}{\partial J_2}J_3\right)^{\frac{1}{2}}} \tag{14}$$

Substituting this into Eq. (12) gives

$$de_{ij}^p = \frac{\sqrt{\frac{3}{2}}}{\frac{2}{3}\sigma_i^p H_p \left(\left(\frac{\partial f}{\partial J_2}\right)^2 2J_2 + \left(\frac{\partial f}{\partial J_3}\right)^2 \frac{2}{3}J_2^2 + 6\frac{\partial f}{\partial J_3}\frac{\partial f}{\partial J_2}J_3\right)^{\frac{1}{2}}} \partial f \frac{\partial f}{\partial s_{ij}} \tag{15}$$

Now that the plastic strain tensor is expressed in terms of the plastic hardening modulus (which will later be related to the Ramberg Osgood equation) we can take $\partial f = \frac{\partial f}{\partial J_2}dJ_2 + \frac{\partial f}{\partial J_3}dJ_3$ and $\frac{\partial f}{\partial s_{ij}}$ becomes $\frac{\partial f}{\partial s_{ij}^p} = \frac{\partial f}{\partial J_2}s_{ij}^p + \frac{\partial f}{\partial J_3}t_{ij}^p$. Where $t_{ij}^p = s_{ik}^p s_{kj}^p - \frac{2}{3}J_2\delta_{ij}$ and the following elastic plastic deviatoric stress invariants have been used $J_2 = \frac{1}{2}s_{ij}^p s_{ij}^p$ and $J_3 = \frac{1}{3}s_{ij}^p s_{jk}^p s_{ki}^p$. Symmetry of the loading process yields $s_{11}^p = s_{22}^p$. s_{33}^p is equal to $-2s_{11}^p$ which is found from $J_1 = s_{ii}^p = 0$. Evaluating the deviatoric invariants and effective plastic stress yields

$$\begin{aligned} J_2 &= \tfrac{1}{2}s_{ij}^p s_{ji}^p = 3s_{11}^{p\,2} \\ J_3 &= \tfrac{1}{3}s_{ij}^p s_{jk}^p s_{ki}^p = s_{11}^p s_{22}^p s_{33}^p = -2s_{11}^{p\,3} \\ dJ_2 &= s_{ij}^p ds_{ij}^p = 6s_{11}^p ds_{11}^p \\ dJ_3 &= -6s_{11}^{p\,2} ds_{11}^p \\ \sigma_i^p &= s_{11}^p\left(9 - \tfrac{4c}{3}\right)^{\frac{1}{2}} \end{aligned} \tag{16}$$

From the Ramberg Osgood equation, $\epsilon_e^p = a(\frac{\sigma_e^p}{b})^{2n+1}$, we have $H_p = \frac{b}{a(2n+1)}(\frac{b}{\sigma_e^p})^{2n}$. For this analysis we take n = 1. Evaluating the first principal plastic deviatoric strain, de_{11}^p, and making all the appropriate substitutions and calculations yields

$$de_{11}^p = \frac{a(27-4c)s_{11}^{p2}\sqrt{3}\left(3-\frac{4c}{9}\right)^{\frac{1}{2}}ds_{11}^p}{2b^3} \tag{17}$$

Integrating gives

$$e_{11}^{p} = \frac{\alpha(27-4c)s_{11}^{ps}}{sb^{s}\sqrt{3}}\left(3-\frac{4c}{9}\right)^{\frac{1}{2}} \tag{18}$$

By substituting Eq. (10) and the Ramberg Osgood equation $e_{11}^{p} = \frac{1}{3}(1+\nu)\epsilon_{i}^{p} = \frac{\alpha\sigma_{i}^{ps}}{2b^{s}}$ into Eq. (18) we can solve s_{11}^{p} in terms of the effective plastic stress

$$s_{11}^{p} = \sigma_{i}^{p}\left[\frac{\sqrt{3}}{(27-4c)\left(3-\frac{4c}{9}\right)^{\frac{1}{2}}}\right]^{\frac{1}{s}} \tag{19}$$

Notice upon substituting $c = 0$, representing simple J_2 plasticity, we obtain

$$s_{11}^{p} = s_{22}^{p} = -\frac{1}{2}s_{33}^{p} = \sigma_{i}^{p}\left[\frac{\sqrt{3}}{27\sqrt{3}}\right]^{\frac{1}{s}} = \frac{\sigma_{i}^{p}}{3} \tag{20}$$

And so our results are consistent with Eq. (11) of Iliushin's theory. This formulation is easily combined with Li's solution of the residual stress for shot peening. When reverse yielding does not occur the residual stresses can be defined as

$$\sigma_{ij}^{r} = \begin{cases} 0 & for\ \sigma_{i}^{e} < \sigma_{s} \\ s_{ij}^{p} - s_{ij}^{e} & for\ \sigma_{s} \le \sigma_{i}^{e} \le \sigma_{s} \end{cases} \tag{21}$$

Or in component form

$$\sigma_{11}^{r} = \sigma_{22}^{r} = -\frac{1}{2}\sigma_{33}^{r} = s_{11}^{p} - \frac{1}{3}\sigma_{i}^{e} \tag{22}$$

s_{11}^{p} is provided in Eq. (19). If reverse yielding occurs and we assume isotropic hardening then the residual stresses have the form

$$\sigma_{11}^{r} = \sigma_{22}^{r} = -\frac{1}{2}\sigma_{33}^{r} = s_{11}^{p} - 2s_{11}^{p} - \frac{1}{3}\Delta\sigma_{i}^{p} \tag{23}$$

Where $\Delta\sigma_{i}^{p} = \frac{k\alpha\Delta\sigma_{i}^{e}}{E} = \frac{k\alpha(\sigma_{i}^{e}-6s_{11}^{p})}{E}$

Discussion and Summary

Figure 1 show how the residual stress varies for different values of c. Shown in figure 2 are two plots of the residual stress calculated from Eq. (22), (23) and (19) for c = 0, which corresponds to the residual stress obtained from Iliushin's theory, and c = -3.375. Published experimental data is used Cao [10]. Miao [9] corrects the data because of material removal that alters the residual stress measurement. It is clear that the predictions provided for c = -3.375 compare much better to experimental residual stress data. Material properties used can be found in [9]. The analysis here indicates that the complex loading process of shot peening interferes with accurate analytical modeling of the residual stresses. A rigorous calculation of the elastic plastic deviatoric stress incorporates the yielding behavior that occurs for complex loading scenarios. This improved deviatoric stress is easily included in Li's residual stress model to allow for a better fit to experimental data.

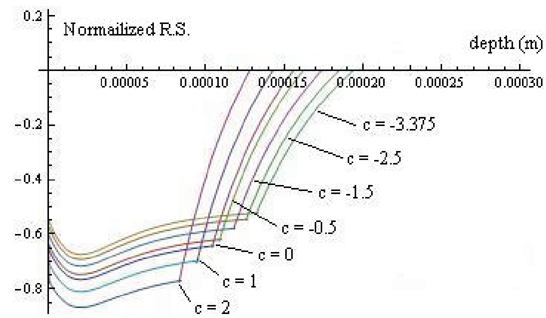

Figure 1. Plots of normalized residual stress as a function of depth for c between 2.0 and -3.375.

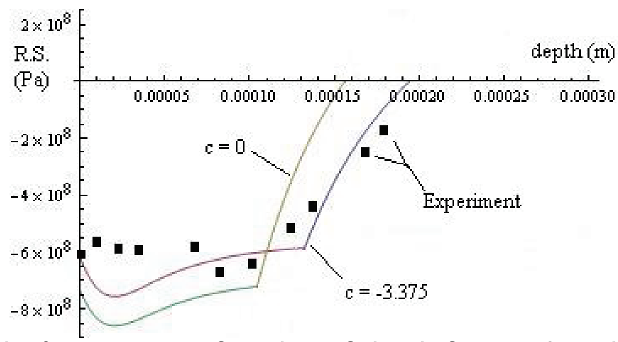

Figure 2. Plots of residual stress as a function of depth for c = 0 and c = -3.375 compared to published experimental data [9, 10]

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