

A METHOD OF DERIVING RESIDUAL STRESS EQUATIONS

J. W. LAMBERT, Purdue University, West Lafayette, Ind.

ABSTRACT

A new method of deriving equations relating the distribution of residual stress to the strains measured during removal of successive layers of material is presented. The method is used to derive Sachs' equations for polar symmetric stresses in circular bars, and also the relation between antisymmetric stresses (left by plastic bending) in round bars and surface strains measured while removing successive concentric circular layers of material from the inside. The method generally leads to an integral equation. In the case of Sachs' equations, the integral equations are reducible by differentiation. In the antisymmetric case, one gets a Volterra integral equation of the first kind which has an approximate solution in terms of Legendre Polynomials, but which is more easily solved by a finite difference method due to Volterra. The antisymmetric case can be easily generalized to give the desired relationship for any prismatical bar.

LIST OF SYMBOLS

- A_R = total cross-section of bar (square inches),
 A_r = cross-section area of bar included in radius r (square inches),
 c = distance from neutral plane to the point of measurement of ϵ_a (inches),
 E = modulus of elasticity (pounds per square inch),
 ϵ_a = polar symmetric part of axial strain (inches per inch),

- ϵ_a' = antisymmetric part of axial strain (inches per inch),
 ϵ_t = measured tangential strain (inches per inch),
 $f(y)$ = unknown antisymmetric stress distribution (pounds per square inch),
 I_r = moment of inertia of cross-section between radii r and R (inches⁴),
 R = outside radius of round bar (inches),
 r = radius of hole drilled in specimen (inches),
 y = coordinate -axis $\perp$ plane of antisymmetry,
 Θ = $\epsilon_t + \nu \epsilon_a$ (inches per inch),
 Φ = $\epsilon_a + \nu \epsilon_t$ (inches per inch),*
 ν = Poisson's ratio (dimensionless),
 ρ = variable of integration $\leq r$ (inches),
 σ_a = polar symmetric axial residual stress (pounds per square inch),
 σ_r = polar symmetric radial residual stress (pounds per square inch),
 σ_t = polar symmetric tangential residual stress (pounds per square inch),
 $\sigma'_{r,t,a}$ = apparent stresses which cause the measured surface strains (pounds per square inch).

INTRODUCTION

The experimental determination of a residual stress distribution is in general a difficult problem. One practical method of observing a state of stress is by means of X-rays. However, the method is restricted to determining stresses on a surface. Alternately, layers of

* In work done by Sachs [2] ** and others λ is used for this quantity.

** Numbers in brackets pertain to references given at the end of the paper.

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the material may be removed to uncover other surfaces in order to determine residual stresses below the original surface. In this case, the change in stress due to removal of the stressed layers on the outside must be calculated.

Mechanical methods of residual stress analysis consist of observing strains which take place in the material when residually stressed sections are removed. Several limitations to this method have been overcome. The difficulty of making the small strain measurements required has been overcome to a great degree by the use of SR-4 resistance strain gages. The problem of removing layers without inducing new residual stresses by the machining operation has received considerable study [1]. The main obstacle to this method at present seems to be that of relating the strains observed to stress which existed in the material which was removed.

Thus far only a few of these relations have been worked out. The best known examples of these are: (1) "Sachs' equations" [2] which relate the observed surface strains to the polar symmetric radial, tangential and longitudinal components of residual stress in round bars, and (2) the relation between longitudinal residual stress and strain (or curvature) measurements in rectangular bars. A few other special cases have been worked out.

In all cases that have been worked out, the method of deriving the relation has been the same. The strain resulting from the removal of a typical layer is related to the stress existing in that layer and then the stress in this layer is "corrected" by calculating the amount this stress has changed by removing previous layers. This line of reasoning is not necessary since the total change in strain after removing any number of layers does not particularly depend upon the stress that was in the last layer removed nor on the thickness of the layer but depends only on all the stress that was contained in all the material which has been removed up to this point. This line of reasoning will be employed in the derivations to be presented, and yields the desired equations more directly.

It is also assumed that the "apparent" stresses which cause the strains that are measured can be superimposed on the residual stresses originally present without dis-

turbing the residual stress distribution. The assumption is valid if the sum of the apparent stress and the residual stress does not exceed the proportional limit stress of the material. This can be checked by calculation after the experiment is performed, if desired.

DERIVATION OF SACHS' EQUATIONS

Use will be made of the following relationships from the theory of elasticity. Tangential stress at the surface of a thick-walled tube due to a uniform internal pressure p_i are given by [3]:

$$\sigma_t' = \frac{2r^2 p_i}{R^2 - r^2}, \quad (1)$$

and from the generalized Hooke's law,

$$\sigma_a' = \frac{E}{1-\nu^2} (\epsilon_a + \nu \epsilon_t) = \frac{E}{1-\nu^2} \Phi, \quad (2)$$

$$\sigma_t' = \frac{E}{1-\nu^2} (\epsilon_t + \nu \epsilon_a) = \frac{E}{1-\nu^2} \Theta. \quad (3)$$

These results are to be applied on the surface of the tube where $\sigma_r' = 0$ and:

$$\begin{aligned} \epsilon_t + \nu \epsilon_a & \text{ is defined as } \Theta, \\ \epsilon_a + \nu \epsilon_t & \text{ is defined as } \Phi. \end{aligned}$$

Now consider that the specimen has been bored out to an arbitrary radius r . A uniform pressure p_i has been removed from the inside boundary due to the removal of material containing polar symmetric residual stresses. At radius r , the p_i removed is equal to the original radial stress at that point before boring started, therefore:

$$\sigma_r = p_i. \quad (4)$$

Substituting from Equations (1) and (3):

$$\begin{aligned} \sigma_r &= \frac{R^2 - r^2}{2r^2} \sigma_t', \\ \sigma_r &= \frac{E}{1-\nu^2} \cdot \frac{R^2 - r^2}{2r^2} \Theta. \end{aligned} \quad (5)$$

To put into form used by Sachs, multiply numerator and denominator by π , giving:

$$\sigma_r = \frac{E}{1-\nu^2} \frac{A_R - A_r}{2A_r} \Theta. \quad (6)$$

This gives the radial residual stress as a function of r and the strains measured at various stages of boring.

For the tangential stress, consider the slug of material of radius r which has been bored out. It, too, was subject to the uniform pressure p_i . The relationship between the tangential stress and the external pressure p_i is obvious from a free body diagram of half the slug. If the slug is split by a plane through its axis and any diameter, the equilibrium of the forces acting gives:

$$r p_i = \int_0^r \sigma_t(\rho) d\rho. \quad (7)$$

Substituting (1) and (3):

$$\sigma_t' = \frac{2r}{R^2 - r^2} \int_0^r \sigma_t(\rho) d\rho = \frac{E}{1-\nu^2} \Theta,$$

$$\text{or } \int_0^r \sigma_t(\rho) d\rho = \frac{E}{1-\nu^2} \frac{R^2 - r^2}{2r} \Theta. \quad (8)$$

Differentiating both sides of the equation with respect to r :

$$\sigma_t(r) = \frac{E}{1-\nu^2} \left[\frac{-R^2 - r^2}{2r^2} \Theta + \frac{R^2 - r^2}{2r} \frac{d\Theta}{dr} \right]. \quad (9)$$

To put into the form used by Sachs multiply the first term in the brackets by π/π and for the second term use:

$$\frac{d\Theta}{dr} = \frac{d\Theta}{dA_r} \frac{dA_r}{dr} = 2\pi r \frac{d\Theta}{dA_r}, \quad (10)$$

$$\sigma_t = \frac{E}{1-\nu^2} \left[(A_R - A_r) \frac{d\Theta}{dA_r} - \frac{A_R + A_r}{2A_r} \Theta \right]. \quad (11)$$

For the longitudinal residual stress, it is assumed that the specimen is long enough so

that plane strain conditions are valid. When the specimen is bored out to a radius r a longitudinal boundary force F is removed from the inner surface of the remaining hollow cylinder.

Considering the slug which was removed:

$$F = \int_0^r \sigma_a(\rho) dA_\rho = 2\pi \int_0^r \sigma_a(\rho) \rho d\rho. \quad (12)$$

F sets up the apparent stresses σ_a' such that:

$$\sigma_a' = \frac{F}{\pi(R^2 - r^2)} = \frac{E}{1-\nu^2} \Phi, \quad (13)$$

from Equation (2). Substituting (12) into (13) gives:

$$2 \int_0^r \sigma_a(\rho) \rho d\rho = \frac{E}{1-\nu^2} (R^2 - r^2) \Phi. \quad (14)$$

Differentiating both sides with respect to r :

$$2r \sigma_a(r) = \frac{E}{1-\nu^2} \left[(-2r\Phi) + (R^2 - r^2) \frac{d\Phi}{dr} \right],$$

$$\text{or } \sigma_a = \frac{E}{1-\nu^2} \left[-\Phi + \frac{R^2 - r^2}{2r} \frac{d\Phi}{dr} \right]. \quad (15)$$

To get the form used by Sachs, proceed as from Equations (9) to (11) with

$$\frac{d\Phi}{dr} = 2\pi r \frac{d\Phi}{dA_r},$$

$$\sigma_a = \frac{E}{1-\nu^2} \left[(A_R - A_r) \frac{d\Phi}{dA_r} - \Phi \right]. \quad (16)$$

DERIVATION OF RELATION FOR ANTISYMMETRIC STRESSES

A round bar is considered which has anti-symmetric longitudinal residual stresses. These stresses might have resulted from bending in the plastic range. The antisymmetric stress may have symmetric stresses superimposed. Successive concentric circular

layers are removed from the inside and the standard strain measurements [4] are made on the outside surface.

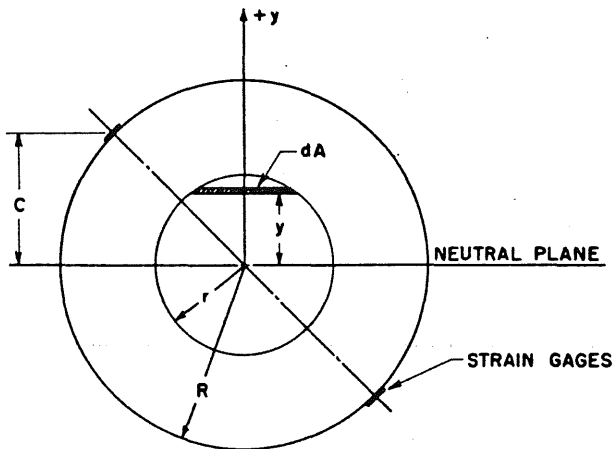


FIGURE 1.

Assume material has been drilled out, out to a radius r . Since this material contained by hypothesis stresses $f(y)$ which were antisymmetric, a moment M has been removed from the inside boundary such that

$$M = \int_{A_r} f(y) y dA.$$

Since $dA = 2(r^2 - y^2)^{1/2} dy$,

$$M = 4 \int_0^r f(y) y (r^2 - y^2)^{1/2} dy. \quad (17)$$

The remainder of the bar will be strained by the removal of M ,

$$e_o = \frac{MC}{EI},$$

$$I = \frac{\pi}{4} (R^4 - r^4),$$

$$\text{Thus } M = \frac{E\pi}{C4} (R^4 - r^4) e_o. \quad (18)$$

Equating (17) and (18):

$$\frac{E\pi}{16C} (R^4 - r^4) e_o = \int_0^r f(y) y (r^2 - y^2)^{1/2} dy. \quad (19)$$

This is a Volterra integral equation of the first kind. The left hand side is an experimentally known function of r since e_o are the antisymmetrical strains measured at various stages r of the boring-out process. The distribution of residual stress $f(y)$ is the unknown. The best method of solution of this equation appears to be that of finite differences due to Volterra and discussed by Richardson [5]. It is to be noted that if $f(y)$ is expressed as a series $f(y) = \sum_n A_n y^n$, where n are even positive integers, the right hand side can be integrated, giving:

$$\frac{E\pi}{16C} (R^4 - r^4) e_o = \sum_k b_k r^k, \quad (20)$$

where the k 's are odd positive integers, and the b_k are known functions of the A_n . Then the series on the right must be matched to the function on the left. This process is simplified by taking $\sum_k b_k r^k = \sum_j c_j P_j(r)$, where the P_j are the even Legendre Polynomials. The orthogonal properties of the P_j facilitate a least square matching of a finite number of terms. However, it has been the author's experience that this method becomes quite tedious and inaccurate as compared to the finite difference method.

A particular advantage of the finite difference method comes from the fact that the bulk of the calculation involves calculating various values of the kernel $y(r^2 - y^2)^{1/2}$. We note that these calculations depend only on the size of the bar used, thus if a number of specimens of the same size are being tested, a table of the values of the kernel can be set up and used for all specimens.

THE MEASUREMENT OF e_o

e_o are antisymmetric longitudinal strains measured at a distance c from the neutral plane. In general, the orientation of the neutral plane will not be known before the experiment. The neutral plane may be deter-

mined if at least two pairs of longitudinal strain measurements are made. The gages in each pair must be mounted diametrically opposite each other, and on a different diameter than those in the other pair. The antisymmetric strain is determined by subtracting the average strain on all gages from the strain on each gage. The remaining strain on each gage is the antisymmetric strain and should be equal and opposite on the diametrically opposed gages. Comparison of the strains on two or more pairs of gages should locate the neutral plane as follows:

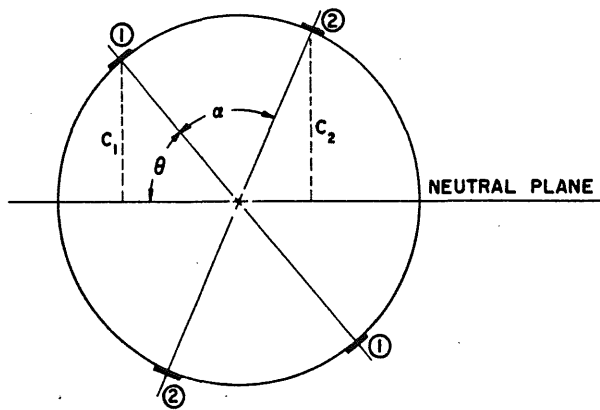


FIGURE 2.

The strain gages 1 and 2 are mounted on diameters which are separated by the known angle α and are at distances c_1 and c_2 from the neutral plane which is at say an angle θ from the diameter of gages 1. It follows immediately that

$$\frac{\sin \theta}{\sin (180 - \alpha - \theta)} = \frac{e_1}{e_2}$$

If α is taken to be 90° , then $\tan \theta = \frac{e_1}{e_2}$. (21)

It should also be noted that Equation (19) can be easily extended to cases of antisymmetric stresses in any prismatical bar. The expression for ϵ in Equation (18) can be recalculated

for other than circular shapes and since this appears in the left hand side of (19), nothing which was said about the solution is changed. The only precaution necessary is that the holes bored to remove material must be bored concentric to the centroid of the cross-section of the bar. The method could also be easily extended to the case where material is removed from the outside of a hollow cylinder.

It is not claimed that this is the most sensitive method of determining antisymmetric stresses. However, the accuracy of the stresses calculated is as good as the solution of Equation (19) which can, mathematically, be made as precise as we wish. It might also be pointed out that any residual stresses induced by the machining operation will be polar symmetric and these will be automatically eliminated when ϵ_0 is calculated.

CONCLUSIONS

The value of this analysis lies in the fact that "Sachs' boring-out method" has been extended. Having performed the experiment on one specimen, we can, using Sachs' equations, determine the radial, tangential and longitudinal polar symmetric residual stresses. By applying the analysis outlined above to the same data one can now evaluate the antisymmetric stresses as well, with no additional experimentation.

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