A New Approach to Plasticity and Its Application to Blunt Two Dimensional Indenters

The classical slip line field solution for a two-dimensional punch is found to give a constraint factor of 2.57 which is too small when the specimen beneath the punch is extensive. A new approach based on elasticity provides a constraint factor of 2.75. The new method of analysis also enables residual stresses to be estimated and indicates that plastic flow occurs not only when the specimen is loaded, but also when it is unloaded. Several details concerning the performance of hardness indenters are explained by the new theory for the first time.

Introduction

The indentation of one body by another has been of engineering interest for a very long time. Brinell [1] introduced the concept of indentation hardness of a material by measuring the mean stress on the surface of a hard sphere when making a permanent impression in a softer metal. Meyer [2] simplified this concept by substituting the area in the plane of the original surface in place of the spherical area of contact in determining the mean indenting stress. Smith and Sandland [3] used a square pyramid having faces intersecting at 136 deg (the so-called Vickers indenter) in place of the sphere in order to provide geometrical similitude for indentations of different depth. Knoop [4] employed an elongated pyramid to provide information concerning anisotropy and to give a more superficial indentation. Several books have been written on the technology of indentation hardness, three of the most famous being those of O'Neill [5], Williams [6], and Tabor [7].

Knowledge of the performance of indenters is not only of interest as a materials test (hardness), but is also of importance to a number of other engineering activities such as:

1. Basic studies of friction wear
2. Shot peening to improve fatigue resistance by inducing residual stresses in surfaces
3. Ball and roller bearing technology
4. The theory of the plowing of asperities or of abrasive grains
5. Improvement of surface finish by burnishing
6. Determining the force required in center punching, staking, or chiseling operations

The theory of indentation has a long history, but a number of details of behavior have never been explained. Prandtl [8] presented the first explanation for the fact that the mean stress on a flat punch (or Brinell or Vickers indenter) is appreciably greater than the uniaxial flow stress of the material as determined in simple compression (Fig. 1). Prandtl reasoned that, since the strains induced by indentation were large compared with elastic strains, the material could be considered to act in a plastic-rigid manner. He proposed the flow field shown in Fig. 2(a) in which a two-dimensional punch (about 10 times as wide in the transverse direction as dimension 2(a)) was forced vertically downward with a force $P$ sufficient to make the material begin to flow plastically. This solution postulated a stationary 90-deg nose which acted as a cutting tool to displace material to the sides and upward as shown. The relation between the mean stress on the punch face to cause plastic flow ($\sigma$), and the plane strain flow stress in uniaxial compression ($S$ from a test as shown in Fig. 1) was found to be

![Fig. 1 Simple uniaxial plane strain compression test](image-url)
Later, Hencky [9] and Hill [10] formalized the procedure used by Prandtl into what has come to be known as the slip line field technique. This approach to large strain plasticity has been very successful, and numerous texts may be consulted concerning details of the method (Hill, [10], Johnson and Mallor [11], Alexander and Brewer [12], and Mulhearn [13]). The slip line field method is only applicable to situations involving plane strain (no flow in one direction—the direction perpendicular to the paper in Fig. 2(a)). It is applied by first assuming a plastic field of stress consisting of straight and curved lines of maximum shear stress. Flow is assumed to follow these lines, and an acceptable slip line field is one for which all velocities are consistent. It is also necessary that the rate of work due to external loads be equal to or greater than the rate of internal energy dissipation for the assumed field. However, force equilibrium is not checked, and the method leads to an upper bound solution for the required external load that is not unique. The best upper bound solution will, of course, be the one predicting the lowest required load.

Hill [10] has presented a second admissible slip line field for the two-dimensional flat punch problem (Fig. 2(b)). This leads to the same value of the ratio \( \frac{p}{S} \) as the Prandtl solution (equation (1)).

As a matter of convenience the quantity \( \frac{p}{S} \) will be referred to as a constraint factor \( C \) in the following discussion. In the slip line field approach the material is constrained to flow in a particular way in order that it satisfy velocity requirements. Therefore, the value of \( C \) derived from the slip line approach will be termed a velocity constraint.

In recent years, careful experimental work has increased our knowledge of the indentation process (for example, see Dugdale [14], Williams and O'Neill [15], Samuels and Mulhearn [16], and Mulhearn [17]). It has been established experimentally that in the Brinell test (which is axisymmetrical as opposed to plane strain) the value of \( C \) is closer to 3 than to 2.57. Also, Samuels and Mulhearn [16] have identified two types of behavior. Sharp wedge-shaped indenters having an included angle less than about 120° give cutting type flow patterns that resemble those predicted by the slip line field approach, while blunt indenters (including angle 120–180°) do not. Samuels and Mulhearn [16] suggest that blunt indenters involve a compression mechanism but do not provide a theory to replace the long standing slip line field approach. Several investigators have reported only minor differences in the value of \( C \) for blunt indenters of different geometries (ball, flat punch, or Vickers pyramid). However, this result has not been explained. Hardness values are also expected to be the same for lubricated and unlubricated blunt indenters, and this lack of independence of constraint factor \( C \) on punch face friction is also not explained when present theory is applied to blunt indenters.

**Experiment**

Figs. 3(a) and 3(b) show deformed grids on meridional planes of plasticine and metal specimens that were loaded by balls in the manner of a Brinell test. Fig. 3(c) shows the extent of the plastic region for both of these tests. The shape of the plastic region for this blunt indenter is seen to best resemble those of the slip line fields shown in Fig. 2. In making both of these Brinell tests care was taken to be sure that considerable material was available beneath the ball so that the lower supporting surface did not influence the results. The plastic boundary shown in Fig. 3(c) strongly resembles a line of constant maximum elastic shear stress beneath a frictionless cylindrical punch (Fig. 4). Such curves may be derived by applying Hertz's theory [18] to the case of a two-dimensional smooth cylindrical indenter.

This suggests that elasticity is not really of negligible consequence as originally assumed by Prandtl and that, instead of considering the material to be plastic-rigid in an extensive situation, it should be considered to be plastic-elastic.

This may be demonstrated in reverse by repeating the experiment of Fig. 3 under nonextensive conditions where the plastic-rigid assumption may be expected to hold. Fig. 5 shows deformed grid patterns produced when a thin layer of plasticine was supported on a steel (rigid) substrate. In Fig. 5(b) the plasticine thickness corresponded to the depth of the Prandtl slip line field \( (h = 1.414a) \), while that in Fig. 5(c) corresponded to the Hill case \( (h = 0.707a) \). The extent of the plastic region is seen to closely resemble the Prandtl and Hill cases, respectively, as given in Fig. 2. It may, therefore, be concluded that a blunt indenter will give a plastic flow corresponding to a slip line field only if there is a sufficiently small quantity of elastic material beneath the punch so that the plastic-rigid assumption holds. Otherwise, the problem must be considered as a plastic-elastic one.

In the analysis which follows, the material being indented is considered to be plastic-elastic, and the rationale employed is as follows. Initially, the applied load is considered to be elastically supported regardless of its size. The volume displaced by the indenter must then be equal to the total decrease in volume in the elastic body due to the elastic compressive stresses developed. Since the Tresca flow criterion will be exceeded by the elastic stresses in the vicinity of the surface, the material will flow plastically until the stresses so developed cause the flow criterion to be satisfied. However, it is assumed that this plastic flow takes place in such a way that the external load is not increased.
Fig. 4 Lines of constant maximum elastic shear stress (nondimensional) beneath a two-dimensional cylindrical punch derived from Hertz's theory. The quantity

\[ M = \frac{2\tau}{p_0} = \frac{2\tau}{2p} \]

where \( \tau \) is the maximum shear stress, and \( p_0 \) is the maximum pressure on the punch face.

Elastic Analysis

Since the two-dimensional punch problem is far simpler than that of the three-dimensional axisymmetrical case, it will be considered first. Fig. 4 shows Hertz lines of maximum elastic shear stress beneath a frictionless cylindrical two-dimensional indenter of extent (t) perpendicular to the paper. The ratio of mean stress (p) to maximum stress (p0) is \( \pi/4 \). The numbers on the lines of constant maximum shear stress shown in Fig. 4 are equal to twice the maximum shear stress divided by the maximum stress on the punch face (\( M = 2\tau/p_0 \)). Fig. 6 is a plot of M versus the nondimensional distance beneath the surface (z/a) along the vertical center line.

By the maximum shear theory, we should expect one of the lines of maximum shear stress in Fig. 4 to be the plastic-elastic boundary for a smooth cylindrical punch. The problem is to select the proper line.

By the principle of St. Venant the stress, a considerable distance from the region of load application (such as at \( 0' \) in Fig. 4), should be independent of the load distribution. Thus, it should be possible to replace the distributed load on the punch face by a concentrated load (P) equal in magnitude to the integrated pressure over the punch face and obtain the same values for maximum shear stress for points such as \( 0' \) in Fig. 4. Also, the shapes of the lines of maximum shear stress, a considerable distance from the punch face, should be independent of the load distribution. Since the constraint factor (C) turns out to be sensitive only to the shapes of the elastic-plastic boundary, a considerable distance from the punch face, the value of C will be essentially independent of that required by the elastic solution. The additional stresses due to plastic flow will be residual elastic ones, and it will require no external work to establish them. After plastic flow has taken place and with the load still present, the same elastic stress field as initially present will be there plus the additional residual stresses associated with plastic flow.

When the load is released, the elastic stress field will collapse and the residual stress system will be all that remains. If the flow criterion is now again exceeded, a second plastic flow must occur. The specimen will finally be left in a state of residual stress that is compatible with the flow criterion.
of load distribution. This is fortunate since otherwise we would have difficulty in predicting the mean stress on the punch face.

In the analysis which follows, two widely different distributions will be used: (a) that of Hertz which pertains to a frictionless cylindrical indenter, and (b) the equivalent concentrated load case which corresponds, as we shall see later, to an indenter with considerable friction on its surface.

Fig. 7 shows the well-known Boussinesq [19] solution for a concentrated line load \((P)\) acting on a semi-infinite elastic body. Principal stress \(\sigma_1\) acting at any point \(A\) on a circle passing through \(O\) will be directed toward \(O\) and will equal

\[
\sigma_1 = \frac{2P}{\pi d t}
\]

where:

- \(P\) = total load
- \(t\) = width of specimen perpendicular to paper
- \(d\) = diameter of circle

The other two principal stresses will be zero, and hence, the circles drawn through \(O\) will be lines of constant maximum shear stress \(\tau\) where

\[
\tau = \frac{1}{\pi} \frac{P}{d t}
\]

Fig. 8 shows the concentrated load situation equivalent to that of a cylindrical punch. The point of application of concentrated load \((P)\) will be some distance \((b)\) within the punch, and stress will be distributed from \(O\) to the punch face as though the punch were a wedge \((BOE)\). Such a loading could be realized in practice by making wedge \(BOE\) from the same material as that beneath \(EE\) (hence, same Young’s modulus), but harder than the material to be indented. The material above line \((WEO)\) will be unstressed for this type of loading and, hence, will play no role.

The plastic solution for a wedge of half angle \(\alpha_0\) subjected to a line load \((P)\) is similar to that for a flat surface. At a point \(A\) (Fig. 8), there will be but one principal stress \(\sigma_1\) directed toward \(O\) and

\[
\sigma_1 = \frac{P}{dt \left(\alpha_0 + \frac{\sin 2\alpha_0}{2}\right)}
\]

The circles will again be lines of maximum shear stress \((\tau)\) and

\[
\tau = \frac{P}{2dt \left(\alpha_0 + \frac{\sin 2\alpha_0}{2}\right)}
\]

In order to apply the principle of St. Venant to this problem, it is necessary that distance \((b)\) be selected such that the stress at a distant point \(O’\) in Fig. 8 be the same when concentrated load \(P\) is equal to the integrated value of the Hertz distributed load. This may be conveniently done by choosing \((b)\) such that the best fit is obtained between the Hertz constant shear stress lines (Fig. 4) and the family of Boussinesq circles (Fig. 8). Such a construction is shown in Fig. 9 where Hertz constant shear stress lines are shown on the left and Boussinesq constant shear stress lines are shown on the right side of the centerline. When this is done, the best value of \(\alpha_0\) is estimated to lie between 69 and 74 deg. We will, therefore, tentatively assume \(\alpha_0\) to be 71.5 deg which corresponds to a value of \(b/a = \cot \alpha_0 = 0.335\).

The advantage of using the equivalent concentrated load solution is that the plastic-elastic boundary is immediately evident. It will be the circle that passes through the edge of the punch at \(E\) in Fig. 8. By the maximum shear theory, the maximum shear stress on the plastic-elastic boundary should be \(S/2\), where \(S\) is the plane strain uniaxial compressive flow stress of the material. From Fig. 9 the value of \(M\) for the Hertz maximum shear stress line corresponding to the Boussinesq circle passing through \(E\) is 0.300. Hence, this particular Hertz line, which passes through point \(D\), will be the plastic-elastic boundary.

The fact that the principle of St. Venant does hold for point \(D\) where the critical Hertz and Boussinesq lines intersect in Fig. 9 may be shown as follows. The maximum shear stress from the wedge solution will be (from equation (5))

\[
\tau_1 = \frac{P_1}{2dt \left(\alpha_0 + \frac{\sin 2\alpha_0}{2}\right)}
\]

while that from the Hertz solution will be

\[
\tau_1 = \frac{P}{2dt \left(\alpha_0 + \frac{\sin 2\alpha_0}{2}\right)}
\]
of max. shear stress at E.\(20\)

Figs. 9 Superposition of concentrated load (right) and Hertz (left) solutions for two-dimensional punch

By St. Venant these two shear stresses should be equal at the same point when the loads are equal. Therefore, letting \(P_1 = P_2\) we may observe how close ratio \(\frac{\tau_1}{\tau_2}\) is to unity for point D in Fig. 9, where

\[
\frac{\tau_1}{\tau_2} = \frac{P_3M}{2\pi d} \left( \alpha_0 + \frac{\sin 2\alpha_0}{2} \right)
\]  

(7)

Referring to triangle \(OED\) in Fig. 9, it is evident that

\[
\frac{d}{a} = \frac{b}{a} + \frac{e}{a} = \cot \alpha_0 + \tan \alpha_0
\]  

(9)

Hence, when \(\alpha_0 = 71.5\) deg

\[
\begin{align*}
\frac{b}{a} &= 0.3346 \\
\frac{e}{a} &= 2.9887 \\
\frac{d}{a} &= 3.3233
\end{align*}
\]

and \(M = 0.300\) (Fig. 6) and thus, from equation (8)

\[
\frac{\tau_1}{\tau_2} = 1.000
\]

By definition,

\[
M = \frac{2\tau}{p_0} = \frac{2\tau}{\frac{4}{\pi} \rho}
\]  

(10)

Since the elastic-plastic boundary has been assumed to pass through points \(E\) and \(D\) in Fig. 9, for the concentrated load case, then from the Tresca flow criterion and equation (10)

\[
M = \frac{S}{4} = \frac{\pi}{4C}
\]  

(11)

or

\[
C = \frac{\pi}{4M}
\]  

(12)

For the assumed value of \(\alpha_0 = 71.5\) deg \(M = 0.300\) and, hence, constraint factor \(C = 2.62\).

This is the answer we are seeking provided the angle \(\psi_0\) has been correctly assumed.

However, the fact that \(\tau_1/\tau_2\) equals unity does not constitute a suitable check on the choice of \(\alpha_0\) since other values in the vicinity of \(\alpha_0 = 71.5\) deg will also give values of \(\tau_1/\tau_2\) close to unity. It further appears that a value of \(\tau_1/\tau_2\) close to unity is a necessary but insufficient check on the assumed value of \(\alpha_0\).

A more discriminating check on \(\alpha_0\) involves the consistency of Fig. 9. If \(\alpha_0 = 71.5\) deg then the elastic-plastic boundary will follow the Hertz line corresponding to \(M = 0.300\) to the point of closest approach to the edge of the punch at \(E\). This corresponds to the intersection with line VT (locus of points of closest approach) in Fig. 9 at \(T\). From this point the elastic-plastic boundary should take the direction of maximum shear stress. Since point \(T\) is very close to point \(E\), we may assume the direction of maximum shear stress at \(T\) to be the same as that at \(E\) which, in turn, will be directed 45 deg from line \(WEO\) in Fig. 9. Thus, the elastic-plastic boundary should intersect the face of the punch at \(E'\).

The shear stress at \(E'\) should correspond to \(S/2\) and the check on the assumed value of \(\alpha_0\) consists of showing that this is consistent with the value of constraint \((C)\) obtained from equation (12). For \(C = 2.62\), \(M = 0.300\), and the maximum stress \(p_0\) at the center of the punch according to Hertz will be

\[
p_0 = \frac{S}{M} = 3.338
\]  

(13)

The vertical scale of normal stress on the punch face \((p)\) in Fig. 9 has been scaled off such that \(p_0 = 3.338\).

If a vertical line is erected from \(E'\), it should intersect the semicircle of punch face pressure distribution at point \(U\) which corresponds to \(p = S\). Then, the pressure on the punch face at \(E'\) will be \(S\), and since there is no friction in the Hertz solution, this will give a value of maximum shear stress at \(E'\) of \(S/2\), which also corresponds to the value indicated from the point of view of the material beneath the punch.

It is thus seen that a value of \(\alpha_0 = 71.5\) deg is consistent with the stress distribution across the punch face and within the deformed material. The assumed value of \(\alpha_0 = 71.5\) deg was, of course, arrived at by trial until points \(E'\) from punch and deformed material considerations coincided.

In the foregoing analysis, stresses on the elastic-plastic boundary obtained from the Hertz distributed load and the equivalent concentrated load cases were observed to be identical some distance from the punch (i.e., at point \(D\) and beyond in Fig. 9) and to differ only slightly in the vicinity of the edge of the punch (points \(E\) and \(E'\) which are equivalent for the two cases close together). It is a convenient simplification to be able to substitute the concentrated load equivalent for the Hertz situation when stresses some distance from the contact surface are involved. The fact that constraint factor \(C\) depends primarily on stress in the plastic-elastic boundary is considerable distance from the punch face accounts for the fact that magnitude of \(C\) is the same for either the distributed (Hertz) or concentrated
But,

\[ d_0 = \frac{b}{\cos^2 \alpha_0} \]  
\[ d = \frac{b}{\cos \alpha} \]  

Therefore,

\[ \sigma_{1A} = \sigma_{1E} \left( \frac{\cos \alpha}{\cos \alpha_0} \right)^{1/2} \]  

This stress may be resolved into normal \( p \) and tangential \( \tau \) values as follows:

\[ p_A = \sigma_{1A} \cos \alpha = \sigma_{1E} \left( \frac{\cos \alpha}{\cos \alpha_0} \right)^{1/2} \cos \alpha \]  
\[ \tau_A = \sigma_{1A} \sin \alpha = \sigma_{1E} \left( \frac{\cos \alpha}{\cos \alpha_0} \right)^{1/2} \sin \alpha \]  

Values of \( p \) and \( \tau \) are given in Table 1 and are shown plotted across the punch face for impending plastic flow in Fig. 10(b). Also shown is the local coefficient of friction

\[ f = \frac{\tau}{p} = \tan \alpha \]  

which varies linearly from a maximum of 2.99 at the edge of the punch to 0 at the center.

Table 1 Values of normal \( (p) \) and tangential \( (\tau) \) stresses across punch face for concentrated load

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \cos \alpha )</th>
<th>( p_b )</th>
<th>( p/S )</th>
<th>( \tau/S )</th>
<th>( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.000</td>
<td>9.67</td>
<td>9.67</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.25a</td>
<td>0.801</td>
<td>8.81</td>
<td>7.06</td>
<td>5.28</td>
<td>0.75</td>
</tr>
<tr>
<td>0.50a</td>
<td>0.557</td>
<td>3.30</td>
<td>1.82</td>
<td>2.74</td>
<td>1.49</td>
</tr>
<tr>
<td>0.75a</td>
<td>0.404</td>
<td>1.65</td>
<td>0.67</td>
<td>1.51</td>
<td>2.26</td>
</tr>
<tr>
<td>a</td>
<td>0.317</td>
<td>1.00</td>
<td>0.32</td>
<td>0.95</td>
<td>2.99</td>
</tr>
</tbody>
</table>

The areas under the \( p \) versus \( x \) curves in Fig. 10(a) (Hertz) and Fig. 10(b) (concentrated load) will, of course, be equal. The peak value of normal stress for the concentrated load case (9.675) is seen to be much higher than that for the Hertz case (3.335).

In the concentrated load case, the punch is elastic and is considered to have the same Young's modulus as the material indented.

It is evident that the two loading conditions considered here differ widely relative to both friction and elasticity of punch. The fact that the constraint value \( C \) is the same for these two extreme types of loading explains why lubrication, as well as the stiffness of the indenter, is unimportant in hardness testing with a blunt indenter.

**Plastic Analysis**

Having established that the plastic-elastic boundary for a two-dimensional blunt punch will be the line corresponding to \( Z = 0.300 \) in Fig. 9 which leads to a constraint factor (\( C \)) of 2.62, we should next consider what happens within the plastic region. This will be done first for a point on the vertical centerline (\( A \) in Fig. 11(a)) and then for a more general point (\( E \)).

The action following the application of the external load may be considered in two steps. In the first, the elastic stress field will be fully developed as load \( P \) is applied, but plastic flow will not have occurred (Fig. 11(b)). The only stress present will be \( \sigma_1 \), the other principal stresses being zero. At points within the plastic zone (such as \( A \) in Fig. 11(a)) \( \sigma_1 \) will exceed \( S \), and flow will occur until a transverse stress \( H_1 \) develops such that

\[ H_1 = \sigma_1 - S \]
The stresses which develop following plastic flow must not satisfy the Tresca flow criterion, but they must develop in such a way that the elastic solution at and beyond the plastic-elastic boundary remains unchanged. The latter condition will be satisfied if the net vertical force at every point remains unchanged before and after plastic flow.

Fig. 11(f) shows the particle at B after the load is applied but before plastic flow has occurred. Stress $\sigma_1$ is the elastic stress at B and will be the only stress present. Fig. 11(g) shows the state of stress with the load still applied but after plastic flow. In accordance with the Tresca flow criterion
\[
\sigma' = H_1 = S
\] while in order that the net vertical load at B remain unchanged
\[
\sigma_1 \cos \alpha = \sigma'_1 \cos \alpha + H_1 \sin \alpha
\]
or
\[
H_1 = \frac{\sigma_1 - S}{1 + \tan \alpha}
\]
\[
\sigma_1' = \frac{\sigma_1 + S \tan \alpha}{1 + \tan \alpha}
\]

If $\alpha = 0$, these stresses reduce to those shown in Fig. 11(c) as they should.

The third stress in Fig. 11(g) will be the mean of the other two in accordance with plane strain. Fig. 11(h) shows the state of stress after the external force $P$ is relaxed but before any additional plastic flow takes place. Stress $H_1$ will remain unchanged, and the stress directed at an angle $\alpha$ to the vertical will be
\[
\sigma_1^* = \sigma_1 - \sigma'_1 = H_1 \tan \alpha
\]
If
\[
H_1 + H_1 \tan \alpha < S
\]
then Fig. 11(h) represents the final state of stress. However, if
\[
H_1 + H_1 \tan \alpha > S
\]
a second plastic flow will occur upon unloading, and Fig. 11(i) will represent the final state of stress. In accordance with the Tresca flow criterion
\[
H_1 - H_1 = S
\]
Fig. 12 shows the elastic-plastic boundary for a loaded punch on the left and the magnitude of stress $\sigma_1$ along the vertical centerline on the right (solid line = concentrated load case, dotted line = Hertz case).

The extent of the secondary plastic flow zone for the concentrated load case is shown by a solid curve in Fig. 12. This curve corresponds to the equation
\[
H_1(1 + \tan \alpha) = S
\]
which may be written as follows by use of equations (24) and (27)
\[
\sigma_1 = 2S
\]
The value of $\sigma_1$ equals $2S$ on the punch face at $x = 0.665a$ (point A in Fig. 12) and at the point marked B on the vertical centerline. The secondary plastic flow boundary shown is sketched between these two points. An estimate of the secondary flow zone for the Hertz case is shown by a dotted curve in Fig. 12. The extent of the secondary plastic flow zone for the concentrated load case is shown by a solid curve in Fig. 12. This curve corresponds to the equation
\[
H_1(1 + \tan \alpha) = S
\]
which may be written as follows by use of equations (24) and (27)
\[
\sigma_1 = 2S
\]
load case) and residual compressive stresses that range from $S$ downward to zero over the remainder of the region that was deformed on loading. The extent of the initial plastic zone ($a_1/a$) will be 2.99, while that for the secondary plastic zone ($a_2/a$) will be about 1.35 (concentrated load case). Such information should be of interest to those engaged in peening to increase fatigue strength and the application of ball and roller bearings.

For the concentrated load situation, the specimen material goes plastic all the way to the corner of the punch. Therefore, the width of indentation observed after unloading will be $(2u)$ in such a case. However, for the Hertz case material will be unloaded at $E$ (Fig. 10(a)) and will not reach the plastic state until well within the punch. Fig. 10(a) shows the stress distribution for the Hertz case. The flow stress will not be reached beyond $E'$ where $OE' = \left[\sqrt{(3.33)^2 - 1/3.33}\right]a = 0.953a$. Thus, the indentation associated with a Hertz distribution of load will be smaller than the loaded area of the punch by about 5 percent.

In a hardness test the value of $C$ will be $\frac{2.62}{0.953} = 2.75$ based on the plastic indentation for a frictionless Hertz type loading.

### Characterization of Constraint

The constraint experienced by a blunt indenter penetrating an extensive specimen may now be characterized physically. It is clearly an elastic constraint as opposed to the velocity constraint associated with a slip line field.

#### Where the Displaced Material Goes

When a blunt indenter penetrates an extensive specimen, there is no upward flow of material. The material displaced downward by the punch must be accommodated by the change in volume that accompanies elastic deformation. We may compute the amount of material that must surround the indenter in order to develop the full elastic constraint by use of this concept.

The change in volume per unit volume of an elastic material is

\[
\frac{\Delta V}{V} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3
\]  

(32)

For a concentrated load on a wedge, there is but one nonzero principal elastic stress ($\sigma_1$) and equation (32) may be written

\[
\frac{\Delta V}{V} = \frac{\sigma_1}{E} \left(1 - 2\gamma\right)
\]  

(33)

where

- $E = $ Young’s modulus of elasticity

- $\gamma = $ Poisson’s ratio

The value of $\sigma_1$ is given by equation (4), and hence the total change in volume for a wedge of half angle $\alpha_0$ will be:

\[
\Delta V = 2 \int_0^{\alpha_0} \int_{\rho_1}^{\rho_2} \frac{P}{\left(\alpha_0 + \sin 2\alpha\right) / 2} \left(1 - 2\gamma\right) \sin \alpha \, d\alpha \, d\rho
\]  

(34)

where $\rho_1$ and $\rho_2$ are initial and final radii measured from the point of application of load $P$ in Fig. 8 (point $O$).

The plastic stresses $H_1$ and $[(\sigma_1 + H_1)/2]$ need not be considered since they cause no change in volume.

After integration

\[
\Delta V = 2P \left(1 - 2\gamma\right) \frac{\sin \alpha_0}{\alpha_0 + \sin 2\alpha_0} (\rho_2 - \rho_1)
\]  

(35)

If the indenter is a wedge of half angle $\theta$, then the volume displaced by the wedge will be

\[
\Delta V = a \theta \cot \theta
\]  

(36)

where $2a$ is the lateral extent of the indentation measured in the plane of the surface, and $t$ is the transverse width of the indenter.

Equating equations (35) and (36), we may solve for $\rho_2/2a$, the size of the specimen required such that the elastic volume change corresponds to the displaced material.

\[
\frac{\rho_2}{2a} = \frac{8}{P/2a} \left(1 - 2\gamma\right) \frac{\sin \alpha_0}{\alpha_0 + \sin 2\alpha_0} + \frac{\rho_1}{2a}
\]  

(37)

For the two-dimensional wedge, $\alpha_0$ has been found to be 71.5 deg, and hence

\[
\frac{\alpha_0 + \sin 2\alpha_0}{2} = 1.549 \sin \alpha_0
\]

The quantity $P/2at = CS$ has been found to be 2.75 for a two-dimensional indenter of the Hertz type.

Other values in equation (37) may be estimated for steel as follows:

- $E = 30 \times 10^6$ psi (for steel)

- $\gamma = 0.3$

- $S = 1,000,000$ psi

- $\theta = 75$ deg

- $\rho_1/2a = 0.5$

and upon substitution into equation (37), we obtain

\[
\frac{\rho_2}{2a} = \frac{(\cot 75) \times 30 \times 10^6}{8(2.75 \times 10^6) (0.4)} (1.549) + 0.5 = 14.6
\]
Thus, to insure no upward flow the specimen should extend about 15 times the extent of the impression (2a) for a two-dimensional indenter. The corresponding value is very much less for an axisymmetric indenter since the elastic stress then varies inversely as the square of p instead of inversely as the first power of p. This is one of the reasons for using an axisymmetric indenter instead of a two-dimensional one, and this point is important when considering the relative merits of the Knoop (7.1 length to width ratio) and the Vickers (approximately axisymmetric) indenters for superficial hardness testing.

Relation to Slip Line Field Solution

The value of constraint factor C is observed to be greater by the method presented here (2.75 for case where width of indentation is determined from plastic impression of blunt wedge or cylindrical indenter) than by the slip line field approach (2.57 for frictionless flat punch). This higher value of constraint will only be experienced if the indenter is blunt and ample material is present beneath the indenter (depth of specimen ≈ 15 times width of indentation, 2a), so that there need be no upward flow. If there is upward flow, then C will decrease and approach the slip line field solution as upward flow increases.

Dugdale [14] has presented an interesting study of the indentation characteristics of two-dimensional wedges having included angles (2θ) ranging from 40 to 140 deg. Smooth carefully lubricated hardened steel wedges were pressed into flat surfaces of highly work hardened mild steel, electrolytic copper and pure aluminum, and the mean stress on the punch (p) measured for different degrees of indentation.

In interpreting these results, Dugdale used the slip line field shown in Fig. 13(a) due to Hill, Lee, and Tupper [20] to predict the mean punch stress p in terms of the flow stress of the material from a torsion test. A frictional stress (τ) was assumed to act on the punch face, and the lip that is formed was determined by assuming that all of the material displaced flows upward. That is, lip angle (φ) was obtained by making area ABC equal to displaced area (COD).

Angle ω was obtained from Mohr's circle diagram for material contacting the punch face (Fig. 13(b)) where

$$\omega = \frac{1}{2} \sin^{-1} \left( \frac{\tau}{k} \right)$$

and τ = shear stress on punch face, and k = flow stress of the material in shear (radius of Mohr's circle).

Angle ψ is clearly (θ + ω – φ) and from the Hencky equation

$$\sigma_1 = 2k(1 + \psi)$$

From Fig. 13(b)

$$\sigma = \sigma_1 - k(1 - \cos 2\omega)$$

and from vertical equilibrium of forces on the wedge (Fig. 13(a)),

$$p = \sigma(1 + \mu \cot \theta)$$

Therefore, from equations (39) to (41)

$$\frac{p}{2k} = \left( \psi + \frac{1 + \cos 2\omega}{2} \right)(1 + \mu \cot \theta)$$

If μ is small, it may readily be shown that

$$\frac{p}{2k} = (1 + \theta - \phi) \left( \frac{1 + \mu \cot \theta}{1 - \mu} \right)$$

and by the maximum shear theory \( S = 2k \), where \( S \) is the plane strain uniaxial shear stress. Therefore,

$$\frac{p}{S} = (1 + \theta - \phi) \left( \frac{1 + \mu \cot \theta}{1 - \mu} \right)$$

Measured values of lip angle (φ) are given in Fig. 14. The curve, which lies far above the experimental points, gives the theoretical values for a nonstrain-hardening material. The experimental curve for steel is seen to intersect the zero axis at \( \theta_0 = 80 \). This means that for semiwedge angles from 80 to 90 deg, there will be no upward flow.

Experimental values of \( p/S = C \) are shown plotted versus wedge angle in Fig. 15 together with the theoretical curve for steel from equation (44) (μ = 0.031). In plotting this curve, the
Fig. 14 Lip angle $\phi$ of impressions made with punches of different wedge angle $\theta$. Solid curve is for theoretical values for nonstrain-hardening material and plotted points are experimental [Dugdale, 14].

Fig. 15 Variation of constraint on smooth wedges indenting highly worked metals with wedge semiangle $\theta$ (data from Dugdale [14]).

Fig. 16 Variation of constraint with wedge angle for different conditions of friction and strain hardening.

Fig. 17 Slip line field solutions with friction (dotted) and without friction (solid).

Fig. 18 Effect of strain hardening on slip line field for a wedge indentation suggested by Dugdale [14].

Theoretical values of $\phi$ from Fig. 14 were used. Also shown in Fig. 15 is a horizontal dotted line at $C = p/S = 2.75$ from the elastic constraint analysis presented here, and angle $\theta_0$ from Fig. 14 (experimental limit for upward flow).

The experimental curve appears to intersect the horizontal dotted line at $\theta = \theta_0$, which indicates that there should be no upward flow for wedges with $2\theta$ between 160 and 180 deg, and for such wedges the elastic constraint 2.75 should pertain.

For indenters with $2\theta < 160$, there will be upward flow, and $C$ will drop as $\theta$ is decreased but not in full accord with slip line field theory. There will be an added constraint over that required by the rigid plastic slip line analysis which is associated with the elastic field that must develop beneath an indenter. There will also be an increase in the observed constraint if there is any strain hardening. The difference between the experimental and slip line curves represents the elastic plus strain hardening effects. Since in this study highly worked metals were used, the strain hardening contribution is believed to be negligible.

The picture presented in Fig. 15 is for a practically frictionless wedge and a material that does not strain harden. Both strain hardening and friction play a major role on the flow constraint, but have a negligible effect on the elastic constraint. This results in a constraint diagram for variable friction and strain hardening such as that shown in Fig. 16. While experimental data to verify this is not available for plane strain indenters, it is available for axisymmetric indenters [Dugdale, 21].

Fig. 17 shows the slip line field solution for zero friction (solid) and for an appreciable amount of friction (dotted). Angle $\psi$ will obviously increase rapidly with $\mu$, as will $p/2k$ (equation (32)), since $\psi$ is dominant in this equation. The effect of strain hardening is essentially the same as that of friction, for as Dugdale [14] has suggested, the slip line field will shift from that shown in Fig. 18 for an ideal nonhardening material to the dotted curve passing through $B$ for a strain hardening material. This will obviously cause a substantial increase in $\psi$ just as an increase in $\mu$ does (Fig. 17).

The effects of both friction and strain hardening are very large for wedge angles causing upward flow (particularly for small values of $\theta$), since small changes in these items cause relatively large changes in the flow pattern and hence in angle $\psi$.

Friction should have a negligible effect on the elastic constraint for a blunt indenter, since the material is displaced downward and not along the surface of the indenter as in the case of wedges of small included angle.
Strain hardening that occurs during indentation should also have a negligible effect on the elastic constraint, since this constraint depends only upon the size and shape of the elastic-plastic boundary and the initial flow stress of the material. The elastic-plastic boundary should be independent of whether strain hardening takes place within the plastic zone since it is derived from purely elastic considerations. Likewise, the value of flow stress for initial plastic flow on the elastic-plastic boundary should not be influenced by strain hardening. Of course, the hardness of a material that has been cold worked before testing will be reflected in a greater indenting force but only in proportion to the increase in initial yield stress caused by the cold work.

The main influence of strain hardening that accompanies indentation with a blunt indenter will be on the strain pattern within the plastic zone, upon the resulting residual stresses left behind in the material, and upon the size of the secondary flow zone relative to the primary one.

We are now in a position to define a blunt indenter as one for which there is no upward flow. The maximum included angle for a blunt indenter will depend upon the amount of material beneath the indenter, upon the friction on the punch face, and upon whether the material strain hardens. For a smooth, well-lubricated plane strain punch operating in an extensive block of nonstrain-hardening material, the limiting wedge half angle for no upward flow \( \theta_b \) will be about as shown in Fig. 15 (\( \theta_b \approx 50 \) deg). An increase in punch friction or strain hardening will move \( \theta_b \) downward to \( \theta'_b \) as shown in Fig. 16, since these changes will postpone upward flow. If the material beneath the indenter is insufficient, this will promote upward flow and hence decrease the critical wedge semiangle \( \theta_b \).

For a hardness test the blunt indenter is obviously superior to a wedge causing upward flow, since it is invariant relative to strain-hardening and friction effects. It is also clear what is being measured—the initial uniaxial flow stress of the material.

Three important rules relating to plane strain hardness indenters are:

1. Use a wedge having an included angle of about 160 deg.
2. The indenter should have a rough (matte) finish and be unlubricated.
3. Insure a depth of material beneath the indenter that is from 10 to 15 times the width of the indentation \( (2a) \).

If these rules are followed for a two-dimensional indenter, there should be negligible upward flow and useful reproducible results should be obtained. The plane strain uniaxial flow stress \( S \) may then be approximated from the mean pressure on the horizontal projected area of the punch \( p \) as follows:

\[
S = \frac{p}{C} = \frac{p}{2.75}
\]

### Concluding Remarks

It has been shown that a blunt indenter is not constrained by flow but instead is elastically constrained when the material beneath the indenter is extensive. This leads to a constraint factor of 2.75 for a two-dimensional blunt indenter instead of the 2.57 based on slip line field theory and a plastic-rigid assumption. The elastic solution presented here provides a valuable insight into what takes place in the plastic zone and enables residual stresses associated with a blunt indenter to be estimated.

Before applying these results to the important engineering problems associated with indenters mentioned at the beginning of this paper, it is important to extend the solutions to axisymmetric indenters. This is necessary because most practical problems involve axisymmetric situations which, in turn, have quite different characteristics than two-dimensional ones. A second paper will, therefore, be devoted to axisymmetric indenters.

### Acknowledgment

The authors wish to acknowledge a grant from the National Science Foundation in support of this research.

### References


Reprinted from the May 1970
Journal of Engineering for Industry

Journal of Engineering for Industry MAY 1970 / 479