On the Plastic Flow Beneath a Blunt Axisymmetric Indenter

A new approach to large strain plasticity problems in which the material is considered to behave in a plastic-elastic fashion, instead of as a plastic-rigid body, is applied to the axisymmetric blunt indenter. The ratio of the mean stress on the punch face to the uniaxial flow stress of the material (constraint factor \( C \)) is found to be 2.82 for an extensive specimen. However, it is shown that a small part of the punch face is elastically loaded, and if the loaded punch area is assumed equal to the size of the plastic impression, then the constraint factor to be used is 3.00 instead of 2.82. This is the value to be used in interpreting the ordinary brinell test. Hardness values are shown to be independent of the degree of friction on the face of a blunt indenter and of the elasticity of the indenter. The amount of material required beneath an axisymmetric blunt indenter may be defined as one which gives no upward flow in a hardness test. Upward flow should be avoided in hardness testing since it causes the mean stress on the punch face to produce a given impression to be sensitive to friction and the tendency of the metal to strain harden. Upward flow may be prevented by use of an extensive specimen relative to the depth of the impression, large indenter angle (160–180 deg), and high indenter friction (rough surface and no lubricant). The flow stress measured by a blunt indenter is that corresponding to the onset of plastic flow. When upward flow is permitted, the flow stress measured by an indentation hardness test will correspond to an appreciable plastic strain which increases as the included angle of the indenter decreases. The quantity measured by an indenter that performs with upward flow is, therefore, quite ambiguous when the material tested strain hardens.

Introduction

In a previous paper (Shaw and DeSalvo [1]), a new approach to large strain plasticity was presented and applied to the two-dimensional blunt indenter. The elastic solutions for two distributions of load (Hertzian and concentrated) on a blunt rigid punch acting on a semi-infinite specimen were considered together to determine the value of constraint pertaining \( C = \text{mean stress on punch face/mean uniaxial flow stress} \), which was found to be 2.62. This value, which is higher than the upper bound value obtained from slip line field theory (2.57), is of entirely different origin.

The new analysis considers the specimen to be a plastic-elastic body instead of plastic-rigid one assumed in the slip line field approach. The new constraint is associated with the force required to establish the elastic field beneath the indenter, while the slip line field constraint is concerned with the force required to cause upward flow of the material beneath the punch.

The new approach to the blunt indenter is useful in estimating residual stresses following indentation and predicts a second plastic flow on unloading, as well as a primary one on loading. It also explains a number of observations concerning the indentation process that have hitherto gone unexplained. However, most practical problems are concerned with axisymmetrical indenters, which have significantly different characteristics from those of...
two-dimensional indenters. The present paper, therefore, extends the new treatment to the axisymmetric indenter.

A number of authors have applied the two-dimensional slip line field solutions of the flat punch of Prandtl [2] and Hill [3] to axisymmetric indenters. Hencky [4] assumed slip line fields similar to those used in plane strain (Prandtl) to obtain an approximate solution for the circular flat punch. In this treatment, Hencky employed the Haar-von Karman [5] hypothesis that the circumferential stress is equal to one of the principal stresses on a meridional plane during plastic flow. Hill [3] has criticized this hypothesis, which is introduced so that the number of equations may be made to equal the number of unknown quantities.

Ishlinsky [6] calculated the actual slip line field around a circular punch using a graphical method and obtained a value for the constraint factor of 2.84. The Haar-von Karman hypothesis was also used in this analysis.

Shield and Drucker [7] employed limit analysis to the solution of a square punch. Using the Tresca yield criterion, they obtained a value of 2.855 for the constraint factor (C) for this case. Levin [8] determined the constraint factor (C = 2.92) for a circular smooth punch loaded uniformly. This upper bound analysis was based on a slip line field that resembled Hill’s for a two-dimensional indenter.

Shield [9] applied the Haar-von Karman hypothesis and Tresca yield criterion to the circular smooth punch. His method essentially follows that of Ishlinsky but employs a numerical approach in place of the graphical method of Ishlinsky. Despite this fact, the value he obtained for the constraint factor (C = 2.845) was remarkably close to that of Ishlinsky (2.84). Shield obtained an admissible velocity field compatible with the stress field, and hence his solution is both an upper and lower bound one and thus represents a complete solution. His slip line field for the circular punch is shown in Fig. 1(a). It should be noted that stress is not uniformly distributed over the punch face as in Levin’s case. Shield also extended the stress field into the rigid region [Fig. 1(b)] and showed that the yield criterion was not violated in so doing.

Lockett [10] has extended the work of Shield to rigid cones of varying cone angle using the same assumptions and techniques. This problem is a more complex one than that of the flat punch, since the shape of the lip that is formed by upward flow is not straight and must be determined by iteration. Values of constraint factor C obtained by Lockett are given by the dotted curve in Fig. 2(a). This curve stops at a semicone angle of 52.5 deg since Lockett’s solution is degenerate for smaller angles. The slip line fields determined by Lockett for semicone angles (θ) of 80 and 52.5 deg are shown in Fig. 2(b). Lockett’s slip line field cannot be extended below θ = 52.5 deg, since angle ψ in Fig. 2(b) would then be negative. A different form of slip line solution is, therefore, needed for semicone angles below 52.5 deg.

Haddow and Danyluk [11] have also extended the work of Shield [9] to the case of rigid cones inserted into prepared matching cavities. In this case, there is no lip. The values of constraint factor (C) obtained by these authors are given by the solid curve in Fig. 2(a) for different values of θ. The value for θ = 90 deg is from Shield [9] who used the same assumptions and approach in his analysis of the flat circular punch as did both Lockett and Haddow and Danyluk. The constraint factors of Shield, Lockett, and Haddow, and Danyluk are summarized in Table 1.

The foregoing literature review indicates that a number of solutions for the flat circular punch are available all based on the Haar-von Karman hypothesis, and all assuming the material to be plastic-rigid. The constraint value obtained by Shield (2.845) is an accurate exact solution for these assumptions, and the slip line field on which it is based is shown in Fig. 1(a).
Experiment

Fig. 3 shows deformed grid patterns on the meridional plane of specimens indented by spheres. The deformed region shown in Fig. 3(c) clearly does not resemble the slip line field of the Shield analysis shown in Fig. 1(a). The difficulty lies in the assumption that the material beyond the deformed region is rigid. In reality, it is elastic and, as has been shown in a previous paper (Shaw and DeSalvo [1]), the elasticity of the specimen plays a major role in determining the shape of the plastic-elastic boundary and, hence, relative to the origin and magnitude of constraint factor (C).

The large difference between the flow patterns that are observed and predicted by slip line theory has been noted by previous authors, but the inadmissible assumption has not been identified, nor has an alternative theory been presented. Samuels and Mulhearn [12] and Mulhearn [13], using etch patterns involving the Frey reagent, showed that the deformation beneath a blunt indenter corresponds to a "radial" compression instead of the cutting type of pattern predicted by the slip line approach. Atkins and Tabor [14, 15] have presented additional evidence for compressive type flow with a blunt indenter.

Williams and O'Neill [16] have shown lines of constant hardness (isoskurs) obtained by making microhardness indentations on the meridional plane for a large indentation. These patterns strongly resemble the etch patterns obtained by Samuels and Mulhearn and have little resemblance to the assumed slip line solutions. However, interpretation of such results are difficult due to the large effects associated with residual stresses induced by the initial indentation. For example, a decrease in microhardness near the center of the impression at the surface is referred to as strain softening, while it is really due to the presence of the important macroresidual stresses to be discussed later in this paper.

With the appearance of the digital computer, it has become possible to solve the large number of linear simultaneous equations associated with a finite element approach to elastic-plastic problems. Examples are to be found in the work of Akyuz [17] and Akyuz and Merwin [18]. However, the emphasis of such studies appears to be upon computational techniques, rather than upon the physics of the problems considered.

In the analysis which follows, the material being indented is considered to be plastic-elastic, and the rationale employed is as follows. Initially, the applied load is considered to be elastically supported. The volume displaced by the indenter must then be equal to the total increase in volume in the elastic body due to the elastic compressive stresses developed, and there will be no upward flow. Since the Tresca flow criterion will be exceeded by the elastic stresses in the vicinity of the surface, the material will flow plastically until the stresses so developed cause the flow criterion to be satisfied. However, it is assumed that this plastic flow takes place in such a way that the external load is not increased beyond that required by the elastic solution. The additional stresses due to plastic flow will be residual elastic ones. After plastic flow has taken place and with the load still present, the same elastic stress field as initially present will be there plus the additional residual stresses associated with plastic flow.

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<tr>
<td>90</td>
<td>2.85 (Shield [9])</td>
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<tr>
<td>80</td>
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<td>2.14</td>
<td>2.45</td>
</tr>
<tr>
<td>52.5</td>
<td>1.94</td>
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<td>2.23</td>
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<tr>
<td>30</td>
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<td>2.21</td>
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Table 1

When the load is released, the elastic stress field will collapse, and the residual stress system will be all that remains. If the flow criterion is now again exceeded, a second plastic flow must occur. The specimen will finally be left in a state of residual stress that is compatible with the flow criterion.

Elastic Analysis

Fig. 4(a) shows lines of maximum elastic shear stress beneath a smooth spherical indenter obtained from a Hertz analysis (see for example Timoshenko and Goodier [19] or Davies [20]). The load distribution is hemispherical, and the ratio of mean ($\bar{p}$) to maximum ($p_\text{m}$) pressure is $1/3$. The numbers on the lines of constant maximum shear stress shown are equal to the maximum shear stress divided by the mean stress on the punch face ($M' = \tau/p$). Fig. 4(b) is a plot of $M'$ versus the nondimensional distance beneath the surface ($z/a$) for the vertical centerline of
Fig. 5 Boussinesq solution for concentrated load $P$ on semi-infinite elastic body

By the maximum shear theory, one of the lines of maximum shear stress in Fig. 4(a) should correspond to the plastic-elastic boundary for fully developed plastic flow. In determining which of these lines to select, and in subsequent calculations, it is convenient to consider a second type of loading on the punch—that of a concentrated load ($P$) that is equal in magnitude to the distributed load.

Fig. 5 shows the Boussinesq solution for a concentrated load ($P$) acting on a semi-infinite elastic body. The stresses acting at a point $A$ having coordinates $r$, $\theta$, and $z$ will be as follows:

$$
\sigma_r = \frac{P}{2\pi} \left[ (1 - 2\nu) \left( \frac{1}{r^2} - \frac{z}{r^2(r^2 + z^2)^{1/2}} \right) - \frac{3z^2}{(r^2 + z^2)^{3/2}} \right] 
$$

$$
\sigma_\theta = -\frac{3P}{2\pi} \frac{z^2}{(r^2 + z^2)^{3/2}} 
$$

$$
\sigma_z = \frac{P}{2\pi} \left[ (1 - 2\nu) \left( -\frac{1}{r^2} + \frac{z}{r^2(r^2 + z^2)^{1/2}} + \frac{z}{(r^2 + z^2)^{3/2}} \right) \right] 
$$

$$
\tau_{rz} = -\frac{3P}{2\pi} \frac{r^2z}{(r^2 + z^2)^{3/2}} 
$$

where $\nu$ = Poisson's ratio. The resultant stress acting on a horizontal plane (as in Fig. 5) will be

$$
S = \sqrt{\sigma_r^2 + \tau_{rz}^2} = -\frac{3P}{2\pi d^2} 
$$

where $d$ is the diameter of the circle shown in Fig. 5. Stress ($S$) will have a constant value for points on a single circle and will be directed toward $(0)$ at all points. The circles of Fig. 5 are not lines of constant shear stress as they were for the two-dimensional case (Shaw and DeSalvo [1]).

Fig. 6 shows the concentrated load situation equivalent to
As an example, assume $\psi_0 = 71$ deg (b/a = $\alpha = 0.344$, and $d/a = 3.249$), then the maximum shear stress at points E and D will be as follows:

$$\tau_D = \frac{3.61P}{2\pi d^2(1 - \cos^3 \psi_0)}$$

(15)

$$\tau_D = \frac{1.6P}{2\pi d^2(1 - \cos^3 \psi_0)}$$

(16)

The maximum shear stress at point D on circle OED is obviously less than that at point E. The point on the vertical centerline (D') having the same maximum shear stress as point E may be found as follows:

$$\tau_D' = \frac{3.61P}{2\pi d^2(1 - \cos^3 \psi_0)} = \frac{1.6P}{2\pi d^2(1 - \cos^3 \psi_0)}$$

(17)

where $d_s$ is the value of $d$ corresponding to point D'.

and

$$\epsilon_s = \frac{d_s}{a} - \frac{b}{a} = 1.16 - 0.344 = 1.816$$

From Fig. 4(b) the value of $M'$ corresponding to this value of $\epsilon_s/a$ is 0.177, and the maximum shear stress at point D' from the Hertz analysis will be

$$\tau_{D'} = \frac{M'P'}{\pi a^2}$$

(18)

If the elastic-plastic boundary is assumed to pass through points E and D' for the concentrated load case, then from the Tresca flow criterion and equation (18)

$$\frac{Y}{2} = \frac{M'P'}{\pi a^2}$$

(19)

or

$$C = \frac{P'}{\pi a^2 Y} = \frac{1}{2M'}$$

(20)

For the assumed value of $\psi_0 = 71$ deg, $M' = 0.177$, and hence the constraint factor $C = 2.82$.

This is the answer we are seeking, provided the angle $\psi_0$ has been correctly assumed.

One check on the assumed value of $\psi_0$ (and hence on the distance concentrated load $P$ is above the plane of loading) may be had by comparing the values of shear stress at point D' form the Hertz and concentrated load solutions. By the principle of Saint-Venant, these two values of stress should be essentially the same when the concentrated load ($P$) and the distributed load ($P'$) are equal.

From equations (17) and (18)

$$\frac{Y}{2} = \frac{M'P'}{\pi a^2}$$

(21)

For the assumed value of $\psi_0 = 71$ deg

$$\frac{\tau_{D'}}{\tau_{D'}} = 1.0034$$
which is obviously close to unity. However, other values of $\psi_0$ in the vicinity of $\psi_0 = 71$ deg give equally satisfactory values for $\tau'_{p'}/\tau_{p'}$, and hence this check on $\psi_0$ is not sufficient.

A second more discriminating check is obtained as follows:

Fig. 7(a) shows lines of constant shear stress (Hertz) together with the assumed value of $\psi_0 = 71$ deg and the corresponding interpolated line of constant shear stress for $M' = 0.177$ is shown by the dashed line. The elastic-plastic boundary will follow this Hertz line of constant shear stress to the point $T$ which is closest to the edge of the punch at $E$. At this point, the elastic-plastic boundary will, in turn, follow the direction of maximum shear stress to the face of the punch at $E'$.

Line $TS$ is the locus of points of closest approach (to point $E$) of successive Hertz lines of maximum shear stress. Point $T$ is the intersection of $TS$ and the particular Hertz line of constant shear stress corresponding to the elastic-plastic boundary for the assumed value $\psi_0 = 71$ deg. The direction of maximum shear stress $E'T$ is obtained by constructing the Mohr's circle diagram (Fig. 7(b)) at point $E$ for the concentrated load solution and assuming the direction of maximum shear stress to be essentially the same at points $T$ and $E$.

The shear stress at $E'$ should correspond to $Y/2$, and the check on the assumed value of $\psi_0$ consists of showing that this is consistent with the value of constraint (C) obtained from equation (20). The value of $C$ corresponding to $\psi_0 = 71$ deg was previously found to be 2.82, and since the maximum Hertz stress ($p_0$) on the punch face is $3/2$ the mean value ($p$), it follows that $p_0 = 3/2 (2.82) Y = 4.23 Y$. The vertical scale of normal stress on the punch face ($p$) in Fig. 7(a) has been laid off such that $p_0 = 4.23 Y$.

The distance $O'E'$ may be computed, since the Hertz load distribution is known to be hemispherical with a peak stress equal to 4.23$Y$. Thus, $OE' = \frac{\sqrt{(4.23 Y)^2 - Y^2}}{4.23 Y}$

The diameter of the impression left behind in the surface will be 3 percent less than $(2a)$ for the Hertz case. If the size of dent left in the surface is used as a measure of the total loaded area, this area will be too low by 6 percent for a Hertz-type load distribution. The true hardness would then be 0.94 times the apparent value. Or, the effective constraint factor would be

$$OE' = 0.97a$$

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Table 2  Values of normal (p) and tangential (τ) stresses across punch face for concentrated load

<table>
<thead>
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<th>x</th>
<th>p/Y</th>
<th>τ/Y</th>
<th>f</th>
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<tr>
<td>0</td>
<td>37.8</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1.00</td>
<td>37.8</td>
<td>0</td>
<td>0</td>
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<tr>
<td>0.25</td>
<td>15.8</td>
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<tr>
<td>1.00</td>
<td>0.42</td>
<td>0.14</td>
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\[
\frac{d}{\cos^2 \psi} = \frac{b}{\cos^2 \psi}
\]  

Therefore,

\[
S_A = S_E \left( \frac{\cos \psi}{\cos \psi_0} \right)^4 \cos \psi
\]

Resolving into normal (p) and tangential (τ) components

\[
p_A = S_E \left( \frac{\cos \psi}{\cos \psi_0} \right)^4 \cos \psi
\]

\[
\tau_A = S_E \left( \frac{\cos \psi}{\cos \psi_0} \right)^4 \sin \psi
\]

The local coefficient of friction (f) will be

\[
f_A = \tan \psi
\]

Values of p, τ, and f are given in Table 2 and Fig. 8(b) for different points across the punch face in terms of the three-dimensional uniaxial flow stress (γ).

The peak value of normal stress for the concentrated load case (37.8γ) is seen to be much higher than that for the Hertz case (4.23γ).

In the concentrated load case, the cone-shaped punch is elastic and is considered to have the same Young’s modulus as the material indented.

It is evident that the two loading conditions considered here differ widely relative to both friction and elasticity of punch. The fact that the same constraint factor (C) is obtained for these two cases is consistent with the observations that lubrication and stiffness of indenter are relatively unimportant in hardness testing with a blunt indenter.

Plastic Action

The left side of Fig. 9(a) shows the elastic-plastic boundary for the concentrated load case shown solid and for the distributed load case (Hertz) shown dotted. The right-hand side of this diagram shows the variation of maximum shear stress along the vertical centerline for the concentrated load case in units of γ, the uniaxial flow stress of the material.

Fig. 9(b) is a Mohr’s circle diagram for the elastic stresses for any point on the vertical centerline. The largest principal stress (σ₁) is directed vertically in the z direction and is 15 times as large as the other two principal stresses, which are equal to each other and tensile in character.

Fig. 9(c) shows the elastic stresses on a particle on the vertical centerline that is within the plastic-elastic boundary, but before plastic flow has occurred. When plastic flow occurs, stresses H₁ will develop in the two transverse directions. These stresses will be equal in accordance with the Haar-von Karman assumption (1909). Flow will continue until the Tresca flow criterion is satisfied; and hence

\[
\sigma_1 - H_1 = \gamma
\]  

The vertical component of force at each point will not change as
plastic flow occurs, because otherwise the external load would change and, hence, the size of the plastic-elastic boundary. The plastic-elastic boundary would then no longer pass through the edge of the punch (E) in Fig. 7.

When the external load \((P)\) is removed, stress \(\sigma_1\) will disappear. The state of stress will be as shown in Fig. 9(e) unless \(H_1\) exceeds uniaxial flow stress \((Y)\). If \(H_1 > Y\), then a second plastic flow will occur on unloading and stress \(H_2\) [Fig. 9(f)] will develop until

\[
H_2 = H_1 - Y
\]

in accordance with the Tresca flow criterion.

The extent of the zone of secondary plastic flow will correspond to points where the maximum shear stress is equal to \(Y\). Thus, the limit line for secondary shear will intersect the vertical centerline at \(A\) in Fig. 9(a) and the remainder of the limit line will be as shown.

The plastic material beneath an axisymmetric indenter will contain residual compressive stresses that are several times the uniaxial flow stress \((Y)\). For example, the maximum lateral residual stress developed at the center of an indentation near the surface will equal \(3.23Y\) for a Hertz-type load distribution.

The extent of the primary zone of plastic flow \((e_t/a)\) will be 1.82 beneath the original surface where \(2a\) is the diameter of the indentation. This value is to be compared with 2.99 for the case of a two-dimensional indenter (Shaw and DeSalvo [1]). The extent of secondary plastic flow \((e_t/a)\) for the axisymmetric indenter is 1.18, which was found to be 1.33 for the two-dimensional case. Thus, both the primary and secondary plastic flow zones penetrate deeper into the material in the case of a two-dimensional indentation.

Fig. 10(a) shows an annealed \(\alpha\) brass specimen (3 in. dia by 2 in. high) that was coated with a photosensitive plastic and ex-
posed to ultraviolet radiation through a negative containing 100 grid lines per inch. After development (removal of exposed plastic), the surface was etched in acid to provide a grid etched into the surface of the specimen. An unetched mating member was securely clamped to the etched one using a massive steel split block, and a 5/16-in. dia tungsten carbide ball was forced into the interface. The unetched mating surface is shown in Fig. 10(b) after deformation. The grid from the etched surface was coined into the second one over a very definite area. This is the area over which the transverse stress \( \sigma_2 \) in Fig. 9(d) is equal to \( Y \) or greater. The limit of the transferred region is, therefore, the locus of the secondary flow zone. The elastic-plastic boundary may be obtained from Fig. 10(a), and this is shown by the solid line in Fig. 10(c) together with a tracing of the line for which \( \sigma_2 = Y \) from Fig. 10(b), shown dotted. The shapes of these two curves are seen to be in excellent agreement with the equivalent analytically derived curves shown in Fig. 9(a).

### Required Specimen Size

In order that the elastic constraint fully develop, it is necessary that sufficient material be located beneath the punch so that the volume displaced may be accommodated elastically. Otherwise there will be some upward flow, and not all of the resistance to penetration will be associated with an elastic constraint. When a thin layer of soft material is located on an essentially rigid substrate, upward flow must occur, and a slip line solution giving rise to a velocity constraint will pertain.

When a brine ball of diameter \( D \) is forced into a surface to a given depth (Fig. 11), the diameter of the impression in the plane of the surface will be \( 2a \), and to a good approximation, the volume of material displaced will be (if \( 2a < 0.4D \))

\[
V_1 = \frac{\pi}{2} \left( \frac{a^4}{D} \right)
\]

For a purely elastic constraint to develop, sufficient material must surround this impression so that the change in volume due to the elastic stresses equals this volume. The change in volume per unit volume in an elastic body is equal to the sum of the principal strains \( (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \). This, in turn, may be expressed in terms of the principal stresses as follows:

\[
\Delta V = \frac{1 - 2\nu}{E} (\sigma_1 + \sigma_2 + \sigma_3)
\]

where:

- \( \nu \) = Poisson's ratio
- \( E \) = Young's modulus of elasticity

The total change in volume due to the elastic stresses may then be obtained by integration

\[
V_1 = 2 \int_0^{\psi_0} \int_{\rho_1}^{\rho_2} \left( \frac{1 - 2\nu}{E} \right) (\sigma_1 + \sigma_2 + \sigma_3)(2\pi \rho^2 \sin \psi) \, d\rho \, d\psi
\]

where \( \rho_1 \) and \( \rho_2 \) are the radii shown in Fig. 11.

It may be shown that the sum of the principal stresses at any point having coordinates \( (\rho, \psi) \) will be as follows for the case of a concentrated load \( P \) acting on a cone of half angle \( \psi_0 \):

\[
\sigma_1 + \sigma_2 + \sigma_3 = -\frac{P}{2\pi \rho^2 (1 - \cos^3 \psi_0)} \times \left[ 3 \sin^3 \psi \cos \psi (1 + \cot^2 \psi) - \cos^3 \psi \sin^2 \psi \right]
\]

Substituting into equation (37) and integrating

\[
V_2 = -2 \left( \frac{1 - 2\nu}{E} \right) P(\rho_2 - \rho_1)(2.95)
\]

Equating \( V_1 \) and \( V_2 \) and solving for \( \rho_2/2a \)

\[
\rho_2 = 47.2 \left( \frac{E}{1 - 2\nu} \right) \left( \frac{2a}{\rho} \right) \left( \frac{1}{p} \right) \left( \frac{\rho_1}{2a} \right)
\]

where

\[
p = P/\pi a^2
\]

As an example consider an indentation in steel where

- \( E = 30 \times 10^6 \) psi
- \( \nu = 0.3 \)
- \( p = 3Y = 300,000 \) psi

The value \( 2a/D \) will normally lie between 0.2 and 0.4 (the latter more conservative value will be used), and a good estimate for \( \rho_2/2a \) = 0.5 (Fig. 11).

Upon substituting these values

\[
\rho_2 = 47.2 \left( \frac{30 \times 10^6}{0.4} \right) (0.4) \left( \frac{1}{0.3 \times 10^6} \right) + 0.5 = 2.62
\]

Thus, to insure no upward flow, the specimen should extend about 2.6 times the extent of the impression for a three-dimensional indenter. The corresponding value was previously found to be much larger (approximately 15) for a two-dimensional indenter (Shaw and DeSalvo [1]).

From equation (41), it is evident that the material properties that influence the amount of material that must surround an indenter for the full elastic constraint to develop are

\[
\frac{E}{(1 - 2\nu)(\rho)} = \frac{E}{(1 - 2\nu)(3Y)}
\]

In Table 3, values of this quantity are given and compared with those for steel.
It is evident that the amount of material required to develop a fully elastic constraint increases as Young's modulus increases but as the hardness of the material decreases. For annealed copper, about 20 times the indentation diameter \((2a)\) will be required if upward flow is to be avoided. Since annealed copper strain hardens extensively and since strain hardening has a strong influence on observed hardness when upward flow is present but has very little influence when there is no upward flow (since the quantity measured with full elastic constraint is the initial yield stress at the plastic-elastic boundary), it is very important to avoid upward flow with such materials if consistent and meaningful hardness values are to be obtained. In addition to having sufficient material beneath the indenter, upward flow may be avoided by use of a very blunt indenter (ball of very large diameter) that has a relatively rough surface and is not lubricated. The experimental results presented in the Discussion support this view.

Two Opposing Indentations

When two opposing indenters act upon a specimen (Fig. 12), there will be no interaction, and the full constraint will be developed when the spacing of the two punches is sufficiently great. It is of interest to find this limiting value of punch spacing for an axisymmetric indenter.

It has previously been shown that for a single punch the distance between the effective point of load application and the lowest point on the elastic-plastic boundary (\(D'\) in Fig. 6) is equal to \(2.16a\). The maximum shear stress at this point (\(D'\) in Fig. 6) will equal \(Y/2\) and

\[
\tau_{D'} = \frac{Y}{2} = \frac{1.6P}{2\pi d_s^2(1 - \cos^2 \psi_a)}
\]

(43)

By definition the constraint factor \((C)\) is

\[
C = \frac{P}{\pi a^2 Y}
\]

(44)

and substituting into equation (43)

\[
\frac{Y}{2} = \frac{0.8 a^2 Y C}{d_s^2(1 - \cos^3 \psi_a)}
\]

(45)

When two punches are present (Fig. 12), the stress at \(D'\) in Fig. 12 will receive a contribution from each, and when the material at this point just goes plastic

\[
\frac{Y}{2} = \frac{0.8 a^2 Y C'}{1 - \cos^3 \psi_a} \left( \frac{1}{d_s^2} + \frac{1}{d_s'^2} \right)
\]

(46)

where \(d_s'\) is the distance from the second load to \(D'\) in Fig. 12.

Equating (45) and (46), we obtain

\[
\frac{C'}{C} = \frac{1}{1 + \left( \frac{d_s'}{d_s^2} \right)^2}
\]

(47)

The punch spacing corresponding to any degree of interaction between the loads may be found from this equation. For example, if the interaction reduces \(C\) by 1 percent, then \(C'/C\) is 0.99 and \(d_s'/d_s^2\) will be 0.1. From Fig. 12, it is evident that

\[
\frac{h}{2a} = \frac{d_s^2 + d_s'^2}{2a} = 2 \frac{b}{2a} = \frac{11d_s^2}{2a} - \frac{b}{a}
\]

(48)

Thus, when the two punches are spaced such that \(h/2a = 11.54\), the load required will be 1 percent less than when the indenters act independently \((h/2a = \infty)\).

Similarly, for a 5 percent reduction in the constraint factor \((C'/C = 0.95)\), the value of \(h/2a\) is found to be 5.56.

This calculation should not be extended beyond this point, because the values of \(b/a\) and \(d_s'/a\) that have been substituted are those obtained from the single load analysis. As the degree of interaction increases, a more complex analysis is required in which \(b/a\) changes with the degree of interaction. In fact, the analysis would become very complex since this then represents a case where both elastic and velocity constraints are operative.

While slip line field solutions exist for plane strain indenters which show that the constraint factor \(C'\) approaches one as \(h/2a\) approaches unity, a parallel treatment is apparently not available for pairs of axisymmetric indenters.

Discussion

Dugdale [23] has studied the indentation characteristics of well-lubricated conical indenters in work-hardened specimens of mild steel, pure copper, and pure aluminum. The flow stress of the material was determined from torsion tests. Fig. 13 shows the variation of the observed constraint factor with cone angle. In accordance with the Tresca flow criterion, the uniaxial flow stress has been taken equal to \(2k\). In these experiments, the specimens were 1 in. square and 0.5 in. deep, while indentations were up to 0.2 in. in diameter. The ratio of depth of specimen to impression diameter was, therefore, as low as three in some in-
stances. While this will provide sufficient material beneath the indenter when \( \theta \) is small and there is substantial upward flow, it will be insufficient to allow a fully developed elastic constraint to be established for blunt indenters. The low values of constraint for steel at cone angles of 120 and 140 deg (Fig. 13) is believed to be due to the specimens being too shallow.

The experimental curve (shown solid) has, therefore, been drawn through the upper points in Fig. 13. This curve is seen to intersect \( C = 3.0 \) at an angle \( \theta_0 \approx 170 \). According to the analysis presented here, the constraint factor \( (C) \) should be 3.0 until upward flow commences, which for a low friction indenter operating in a nonstrain hardening material will be for values of \( \theta \) from about 170 to 180 deg. As the cone angle drops below 170 deg the value of constraint should fall, but not so rapidly as a theory based on upward flow would indicate, since there would be an elastic contribution to the total constraint (Shaw and DeSalvo, [1]).

Also shown in Fig. 13 are the values predicted by Shield [9], Lockett [10], and Haddow and Danyluk [11]. These curves also plotted in Fig. 2 are seen to be in poor agreement with the experimental data. While the Haddow and Danyluk curve fits the data better, it is for the case where no lip is present due to upward flow. The lip that is formed increases as \( \theta \) decreases which accounts for the divergence of the two theoretical curves as \( \theta \) decreases.

In order to check the Haddow and Danyluk curve in Fig. 13, specimens should be prepared by machining a conical pattern in the surface that precisely fits the indenter. If the mean punch pressure \( (p) \) required to initiate plastic flow is then measured, the values should be in better agreement with theory. When Haddow and Danyluk [11] performed such tests the data given in Table 4 were obtained. The experimental values given have been corrected for friction by dividing by \( 1 + \mu \cot \theta \), where \( \mu = 0.06 \).

<table>
<thead>
<tr>
<th>Cone semiangle ((\theta)) deg</th>
<th>Mean Stress on Cone ((p)), psi</th>
<th>( C = \frac{p}{Y} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 45 )</td>
<td>76,600</td>
<td>67,822</td>
</tr>
<tr>
<td>( 60 )</td>
<td>78,100</td>
<td>74,420</td>
</tr>
<tr>
<td>( 75 )</td>
<td>79,500</td>
<td>79,100</td>
</tr>
<tr>
<td>( 90 )</td>
<td>85,300</td>
<td>86,490</td>
</tr>
</tbody>
</table>

The experimental values are now seen to be greater than the theoretical ones (except for \( \theta = 90 \) deg where the theory presented here is to be preferred over the slip line solution used by Shield, since for \( \theta = 90 \) deg there will be no upward flow). This is probably due to the fact that, for all values of \( \theta \) other than \( \theta = 90 \) deg, the origin of the constraint will be mixed, being partly due to upward flow and partly of elastic origin. Ignoring the elastic component of constraint will give a predicted value that is too low which corresponds to what is observed.

Dugdale [24] has presented similar data for square pyramidal indenters operating in cold-worked specimens of mild steel, pure copper, and silicon aluminum. The Steel and copper specimens had the same dimensions as before (1 in. square by 0.6 in. deep), while the aluminum specimens were 1½ in. cubes. Flow stress values in shear \( (k) \) were determined from torsion tests, and by the Tresca flow criterion, the uniaxial flow stress was taken to be 2k.

Fig. 14 shows observed constraint factors versus the pyramid semiangle \( \theta \) for: (a) carefully lubricated smooth indenters, and (b) for indenters that had been roughened by etching in nitric acid. The aluminum points are seen to lie distinctly above those for steel or copper in Fig. 14(a) due to the greater depth of the aluminum specimens. The dotted curve drawn through the aluminum points is believed to better represent the constraint as was expected with specimens of sufficient depth to allow the elastic constraint to fully develop.

Fig. 15 shows the size of lip formed during upward flow for each of the experiments of Fig. 14(a). Upward flow is seen to cease at an angle \( \theta \) of about 75 deg for the aluminum specimens, but not until \( \theta \approx 85 \) deg for the copper specimens.

The dotted curve for the low friction indenters in Fig. 14(a) is seen to intersect the \( C = 3.0 \) line at an angle \( \theta_0 \approx 75 \) deg. There will be no upward flow for values of \( \theta > 75 \) deg in this case. Thus, a smooth indenter operating on cold-worked materials may be considered blunt (no upward flow) when \( \theta \) is 150 to 180 degs.

The dotted curve for the high friction situation in Fig. 14(b) intersects the \( C = 3.0 \) line at \( \theta = 50 \) deg. Thus, a high friction indenter operating on a cold-worked material may be considered blunt when \( \theta \) is as low as 100 degs.

The constraint factor is seen to become more sensitive to the angle of the indenter and the degree of friction as the angle of the indenter decreases. For a pyramid having \( \theta = 30 \) deg, the constraint factor \( C \) is seen to vary from about 1.8 when the friction is low to 3.6 when the friction is high.

Dugdale [25] has given square pyramid data for annealed copper. These results are shown plotted in Fig. 16(a) for indenters of low and high friction. In this case, constraint factors are now shown due to the large change in torsional flow stress \( (k) \) with...
shear strain [Fig. 16(b)], which makes it uncertain as to the value to be used when finding $C = p/2k$ when there is upward flow. Therefore, observed values of $p$ are shown plotted against $\theta$ in Fig. 16. The value of $p$ is seen to rise as $\theta$ decreases, particularly when friction is high.

Atkins and Tabor [15] have also studied the indentation characteristics of conical indenters. Specimens of pure copper and mild steel were tested at different degrees of strain hardening ranging from the fully annealed state to natural strains as high as 1.39. All copper specimens initially measured $\frac{3}{4}D$ by 1 in. high. Cold work was introduced by reducing the height of the specimens in uniaxial compression. The fully work-hardened copper specimens had a height of but $\frac{1}{16}$ in. The steel specimens had initial dimensions of $D$ by $\frac{3}{16}$ in., and the most highly worked specimens had a final height of but $\frac{1}{16}$ in.

The constraint factor ($C$) for the most heavily worked specimens (both steel and copper) is shown plotted in Fig. 17 against cone angle ($2\theta$). In determining these values of $C$, Atkins and Tabor took the flow stress to be that corresponding to the cold-worked material before indentation. No additional work hardening was considered to occur during indentation of the heavily worked material. Fig. 17 corresponds to results of Dugdale shown in Fig. 13. Both curves have a minimum, and anomalously low values of $C$ are observed for large values of $2\theta$. The low values of $C$ for large values of $2\theta$ are believed due to the specimen height being too small to allow a fully developed elastic constraint to be produced. The difference in the magnitude and location of the minimum values of $C$ in Figs. 13 and 17 is believed due to differences in lubrication (friction) specimen height relative to indentation diameter, degree of work hardening, and the method employed in determining the flow stress ($Y$) used in finding the constraint factor ($C = p/Y$). The basic flow curves were obtained from torsion tests in the case of Fig. 13, but from compression tests in Fig. 17. Also, a correction for friction was made in the case of Fig. 13, but not in the case of Fig. 17.

However, both Figs. 13 and 17 appear consistent with the new theory, when the fact that the specimens were too shallow for blunt indenters is taken into account.

Atkins and Tabor [15] also found an increase in $C$ with decrease in degree of previous work hardening, particularly for low values of $2\theta$.

From the foregoing discussion, it is apparent that the constraint factor for an axisymmetric indenter will vary with the indenter angle ($2\theta$), friction, and degree of strain hardening as shown diagrammatically in Fig. 18. The advantage of using a blunt indenter (no upward flow) is that it is essentially invariant relative to indenter friction and the tendency for the metal to work harden. Upward flow may be prevented by use of a rough indenter, no lubricant, a large indenter angle ($2\theta$), and an extensive specimen. Another advantage of an indenter that gives no upward flow is that the flow stress being measured is obvious. This corresponds to the strain in the elastic-plastic boundary. When there is upward flow, the flow stress involved in the constraint factor is that corresponding to the mean strain in the zone of flow, which increases significantly as the wedge angle decreases (Atkins and Tabor [15]).
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References